TENSOR PRODUCT VARIETIES, PERVERSE SHEAVES, AND STABILITY CONDITIONS

YIQIANG LI

Abstract. We show that the space spanned by the class of simple perverse sheaves in [Zh08] without localization is isomorphic to the tensor product of a Verma module with a tensor product of irreducible integrable modules of the quantum enveloping algebra associated with a graph. Under the isomorphism, the simple perverse sheaves get identified with the canonical basis elements of the tensor product module. We also show that the two stability conditions coincide in the localization process in [Zh08], by using supports and singular supports of complexes of sheaves, respectively.

1. Introduction

Let $\mathbf{U}$ be the quantum enveloping algebra associated with a graph Γ without loops. Let M_λ, T_λ and V_λ be the Verma module of $\mathbf{U}$ of dominant highest weight λ, its maximal submodule and its simple quotient, respectively.

In his paper [Zh08], Zheng gives a geometric realization of the tensor product $V_{\lambda^\bullet}$ of irreducible integrable representations of $\mathbf{U}$. In the paper, Zheng defines a class $\mathcal{P}$ of simple perverse sheaves on the frame representation varieties of an oriented graph whose underlying graph is Γ. He then shows that, after localization, the space spanned by the class $\mathcal{P}$ is isomorphic to the tensor product $V_{\lambda^\bullet}$ and $\mathcal{P}$ is corresponding to the canonical basis of $V_{\lambda^\bullet}$.

The localization process eliminates extra elements in $\mathcal{P}$ in order to obtain $V_{\lambda^\bullet}$. This is similar to the process of obtaining Nakajima's quiver varieties from Lusztig's quiver varieties in [N94]. The stability condition used in the localization process in [Zh08] utilizes the notion of the support of a complex of sheaves. On the other hand, the notion of the singular support of a complex of sheaves can be used to define the stability condition in the localization process, which is more global because it does not involve the orientation of the given graph. It is not clear if the two stability conditions for the localization process by using the notions of support and singular support, respectively, are equivalent to each other. This question leads us to study the class $\mathcal{P}$ without taking the localization process.

It turns out that the space spanned by the class $\mathcal{P}$ without localization is isomorphic to the $\mathbf{U}$-module $M_0 \otimes V_{\lambda^\bullet}$ and, under this isomorphism, the set $\mathcal{P}$ becomes the canonical basis (or global crystal base) of $M_0 \otimes V_{\lambda^\bullet}$ defined in [L93]. Along the way to prove the above result, we show that the singular supports of the simple perverse sheaves in $\mathcal{P}$ are contained in certain varieties $\mathbf{\Pi}$ and the irreducible components of which have a crystal structure isomorphic to the tensor product crystal of the canonical basis of M_0 with the canonical

Date: December 15, 2011.

1991 *Mathematics Subject Classification.* 17B37; 14F43.

Key words and phrases. Tensor product variety, perverse sheaf, singular support, canonical basis, crystal basis, projective module in category $\mathcal{O}$, stability condition.

basis of $V_{\lambda^\bullet}$. The quotients of the subvarieties of stable points in $\mathbf{\Pi}$ by certain algebraic groups are Malkin and Nakajima's tensor product varieties ([N01, M03]). As a consequence, the two stability conditions of localization are the same because they both characterize the submodule $T_0 \otimes V_{\lambda^\bullet}$. Our arguments also produce a new proof of the fact that the space spanned by the class $\mathcal{P}$, after localization, is isomorphic to the module $V_{\lambda^\bullet}$ and the elements in $\mathcal{P}$ survived after localization are the canonical basis elements of $V_{\lambda^\bullet}$.

A slight modification of the definition of the class $\mathcal{P}$ gives rise to a geometric realization of the tensor product of a Verma module of dominant integrable highest weight with $V_{\lambda^\bullet}$ and its canonical basis.

Note that the modules $M_0 \otimes V_{\lambda^\bullet}$ and their variants are projective objects in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of $\mathbf{U}$. It is interesting to find a natural way to single out all indecomposable projective objects in category $\mathcal{O}$ from these modules.

Contents

2. Preliminaries

2.1. **Quantum groups.** Let Γ be a graph without loops. This is meant to say that we are given two finite sets I and H, together with three maps $^-\colon H \to H$, $','' \colon H \to I$, such that

$$\bar{\bar{h}} = h, \quad (\bar{h})' = h'', \quad \text{and} \quad h' \neq h'', \quad \forall h \in H.$$

Let $\mathbf{Y} = \mathbb{Z}[I]$, $\mathbf{X} = \mathrm{Hom}(\mathbb{Z}[I], \mathbb{Z})$, and $(,) : \mathbf{Y} \times \mathbf{X} \to \mathbb{Z}$ be the canonical pairing. Let $\alpha_i \in \mathbf{X}$, for $i \in I$, be the elements defined by $(i, \alpha_j) = 2\delta_{ij} - \#\{h \in H | h' = i, h'' = j\}$. The datum $(\mathbf{X}, \mathbf{Y}, (,), \{i \in \mathbf{Y} | i \in I\}, \{\alpha_i | i \in I\})$ is the so-called simply connected root datum.

The map $i \mapsto \alpha_i$ defines an inclusion $\mathbb{Z}[I] \hookrightarrow \mathbf{X}$. If $\nu \in \mathbb{Z}[I]$, we write $\nu \in \mathbf{X}$ to represent its image. Let

$$\mathbf{X}^+ = \{\lambda \in \mathbf{X} | (i, \lambda) \geq 0 \quad \forall i \in I\},$$

be the set of dominant weights.

Let $\mathbf{U}$ be the quantum (enveloping) algebra associated with the above simply connected root datum. This is an associative algebra over $\mathbb{Q}(v)$, the field of rational functions over $\mathbb{Q}$, with generators E_i, F_i, K_i and K_i^{-1} for any $i \in I$ and subject to the following relations:

$$\begin{aligned}
&(\mathbf{U}a). \quad K_0 = 1, \quad K_i K_{-i} = 1, \quad K_i K_j = K_j K_i,\\
&(\mathbf{U}b). \quad K_i E_j = v^{(i,j)} E_j K_i, \quad K_i F_j = v^{-(i,j)} F_j K_i,\\
&(\mathbf{U}c). \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v - v^{-1}},\\
&(\mathbf{U}d). \quad \sum_{p=0}^{1-(i,j)} (-1)^p \left[\begin{smallmatrix} 1-(i,j) \\ p \end{smallmatrix} \right] E_i^p E_j E_i^{1-(i,j)-p} = 0, \quad \forall i \neq j \in I,\\
&(\mathbf{U}e). \quad \sum_{p=0}^{1-(i,j)} (-1)^p \left[\begin{smallmatrix} 1-(i,j) \\ p \end{smallmatrix} \right] F_i^p F_j F_i^{1-(i,j)-p} = 0, \quad \forall i \neq j \in I,
\end{aligned}$$

where we use the following notations

$$[s] = \frac{v^s - v^{-s}}{v - v^{-1}}, \quad \forall s \in \mathbb{Z}; \quad [s]^! = [s][s-1]\cdots[1], \quad \left[\begin{smallmatrix} s \\ t \end{smallmatrix} \right] = \frac{[s]^!}{[t]^![s-t]^!}, \quad \forall s \geq t \in \mathbb{N}.$$

The algebra $\mathbf{U}$ is equipped with a Hopf algebra structure whose comultiplication is defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad \forall i \in I.$$

Let $\mathbf{U}^-$ be the negative part of $\mathbf{U}$, i.e., the subalgebra of $\mathbf{U}$ generated by F_i for any $i \in I$. Let $\mathbf{U}^+$ be the positive part of $\mathbf{U}$ generated by the elements E_i for any $i \in I$. Let $\mathbf{U}^0$ be the zero part of $\mathbf{U}$ generated by the elements $K_{\pm i}$ for any $i \in I$. Then we have

$$\mathbf{U} = \mathbf{U}^+ \otimes \mathbf{U}^0 \otimes \mathbf{U}^-,$$

as vector spaces. We also set $\mathbf{U}^{\geq 0} = \mathbf{U}^0 \otimes \mathbf{U}^+$, the Borel subalgebra of $\mathbf{U}$.

Note that $\mathbf{U}^-$ has an $\mathbb{N}[I]$-grading by defining $\deg(F_i) = i$ for any $i \in I$. Let $\mathbf{U}^-_\nu$ be the subspace in $\mathbf{U}^-$ consisting of the homogeneous elements of degree ν. We have

$$\mathbf{U}^- = \oplus_{\nu \in \mathbb{N}[I]} \mathbf{U}^-_\nu.$$

On $\mathbf{U}^-$, we can define a unique Verma module structure of $\mathbf{U}$ with highest weight $\lambda \in \mathbf{X}$ such that

$$K_i(x) = v^{(i, \lambda - \nu)} x, \quad F_i(x) = F_i x, \quad \text{and} \quad E_i(1) = 0, \quad \forall x \in \mathbf{U}^-_\nu, i \in I.$$

We write M_λ for $\mathbf{U}^-$ together with this $\mathbf{U}$-module structure. Let T_λ and V_λ be the maximal submodule and simple quotient of M_λ. If $\lambda \in \mathbf{X}^+$, then T_λ is the left ideal of $\mathbf{U}^-$ generated by the elements $F_i^{(i,\lambda)+1}$ for any $i \in I$. We have a short exact sequence of $\mathbf{U}$-modules:

$$0 \to T_\lambda \to M_\lambda \to V_\lambda \to 0. \tag{1}$$

The modules T_λ and V_λ are compatible with the grading on M_λ. In other words,

$$T_\lambda = \oplus_{\nu \in \mathbb{N}[I]} T_{\lambda,\nu}, \quad T_{\lambda,\nu} = T_\lambda \cap \mathbf{U}^-_\nu \quad \text{and} \quad V_\lambda = \oplus_{\nu \in \mathbb{N}[I]} V_{\lambda,\nu}, \quad V_{\lambda,\nu} = \mathbf{U}^-_\nu / T_{\lambda,\nu}.$$

For any two $\mathbf{U}$-modules M_1 and M_2, we can define on the tensor product $M_1 \otimes M_2$ a $\mathbf{U}$-module structure via the comultiplication of $\mathbf{U}$. Let $\mathbb{Q}(v)_\lambda$ be the one dimensional $\mathbf{U}^{\geq 0}$ module such that $E_i(f) = 0$ and $K_{\pm i} f = v^{(\pm i, \lambda)} f$ for any $f \in \mathbb{Q}(v)$ and $i \in I$. The Verma module M_λ is the induced module $\mathbf{U} \otimes_{\mathbf{U}^{\geq 0}} \mathbb{Q}(v)_\lambda$. We have the following Frobenius property:

$$\mathrm{Hom}_{\mathbf{U}}(M \otimes M_\lambda, N) \simeq \mathrm{Hom}_{\mathbf{U}^{\geq 0}}(M \otimes \mathbb{Q}(v)_\lambda, N). \tag{2}$$

Let $\mathbb{A}$ be the subring of Laurent polynomials in $\mathbb{Q}(v)$. Let $F_i^{(n)} = F_i^n / [n]^!$. The $\mathbb{A}$-form ${}_{\mathbb{A}}\mathbf{U}^-$ of $\mathbf{U}^-$ is the subalgebra of $\mathbf{U}^-$ over $\mathbb{A}$ generated by the elements $F_i^{(n)}$ for $i \in I$ and $n \in \mathbb{N}$.

The $\mathbb{A}$-form ${}_{\mathbb{A}}\mathbf{U}$ of $\mathbf{U}$ is defined to be the subalgebra of $\mathbf{U}$ over $\mathbb{A}$ generated by $F_i^{(n)}$, $E_i^{(n)}$ and $K_{\pm i}$ for any $i \in I$ and $n \in \mathbb{N}$. Here $E_i^{(n)}$ is defined in a similar way as $F_i^{(n)}$.

The $\mathbb{A}$-form ${}_{\mathbb{A}}\mathbf{U}^-$ is compatible with the grading on $\mathbf{U}^-$, i.e., ${}_{\mathbb{A}}\mathbf{U}^- = \oplus_{\nu \in \mathbb{N}[I]} {}_{\mathbb{A}}\mathbf{U}^-_\nu$, ${}_{\mathbb{A}}\mathbf{U}^-_\nu = {}_{\mathbb{A}}\mathbf{U}^- \cap \mathbf{U}^-_\nu$.

The $\mathbb{A}$-form ${}_{\mathbb{A}}\mathbf{U}^-$ is also invariant under the action of ${}_{\mathbb{A}}\mathbf{U}$ on M_λ, denoted by ${}_{\mathbb{A}}M_\lambda$. Similarly, we can define the $\mathbb{A}$-forms, ${}_{\mathbb{A}}T_\lambda$ and ${}_{\mathbb{A}}V_\lambda$, of the modules T_λ and V_λ, respectively. Thus we have the following short exact sequence

$$0 \to {}_{\mathbb{A}}T_\lambda \to {}_{\mathbb{A}}M_\lambda \to {}_{\mathbb{A}}V_\lambda \to 0. \tag{3}$$

2.2. **Crystals.** We recall from [KS97] the definition of a crystal. By definition, a crystal is a set B equipped with five maps:

$$\begin{aligned} &\mathrm{wt} : B \to \mathbf{X}, \\ &\varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \\ &\tilde{e}_i : B \to B \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i : B \to B \sqcup \{0\}, \end{aligned} \tag{4}$$

and subject to the following axioms.

(C1) $\varphi_i(b) = \varepsilon_i(b) + (i, \mathrm{wt}(b))$.

(C2) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i,\ \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \quad \text{if } b, \tilde{e}_i b \in B$.

(C2') $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i,\ \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \quad \text{if } b, \tilde{f}_i b \in B$.

(C3) $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$, $\quad$ for $b, b' \in B$ and $i \in I$.

(C4) $\tilde{e}_i b = \tilde{f}_i b = 0, \quad$ if $\varphi_i(b) = -\infty$.

By convention, we set $\mathrm{wt}_i(b) = (i, \mathrm{wt}(b))$.

The tensor product $B_1 \otimes B_2$ of the two crystals B_1 and B_2 is defined as follows. As a set, $B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$. The five maps on $B_1 \otimes B_2$ are defined by

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$
$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)), \quad \varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1) + \mathrm{wt}_i(b_2), \varphi_i(b_2)),$$
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{otherwise}, \end{cases} \quad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{otherwise}. \end{cases}$$

A crystal morphism ψ from B_1 to B_2 is a map $\psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ satisfying the following conditions:

(5) $\psi(0) = 0$,

(6) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b),\ \varepsilon_i(\psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\psi(b)) = \varphi_i(b)$,

(7) if $\tilde{f}_i(b) = b'$ for $b, b' \in B_1$ and $i \in I$ and $\psi(b), \psi(b') \in B_2$, then we have $\tilde{f}_i(\psi b) = \psi(b')$.

A *strict* crystal homomorphism is a crystal morphism commuting with the maps $\tilde{e}_i$ and $\tilde{f}_i$.

A strict crystal *isomorphism* is a bijective strict crystal homomorphism.

We denote by $B(\lambda)$ the crystal associated with the crystal base of the irreducible integrable representation V_λ of highest weight λ. We also denote by $B(\lambda, \infty)$ the crystal associated with the crystal base of the Verma module M_λ of highest weight λ.

3. A class of U-modules

3.1. **Verma module structures on $\mathbf{U}^-$.** In this section, we write θ_i for the element F_i in $\mathbf{U}^-$ to avoid confusion with the action F_i. Recall that the algebra $\mathbf{U}^-$ has a Verma module structure of highest weight $\lambda \in \mathbf{X}$ defined by

$$K_i(x) = v^{(i, \lambda - |x|)} x, \quad E_i(1) = 0 \quad \text{and} \quad F_i(x) = \theta_i x, \tag{8}$$

for any $x \in \mathbf{U}^-$ homogeneous, and $|x|$ denotes its degree. Note that the commutator relation can be rewritten as

$$E_i(\theta_j x) = \theta_j E_i(x) + \delta_{ij} [(i, \lambda - |x|)] x, \quad \forall x \in \mathbf{U}^- \text{ homogeneous}, i, j \in I. \tag{9}$$

Recall from [L93], there is a unique $\mathbb{Q}(v)$-linear map

$${}_i\bar{r} : \mathbf{U}^- \to \mathbf{U}^-$$

defined by

$$ {}_i\bar{r}(1)=0, \quad {}_i\bar{r}(\theta_j)=\delta_{ij}, \quad {}_i\bar{r}(xy)={}_i\bar{r}(x)y+v^{(i,-|x|)}x\,{}_i\bar{r}(y), \quad \forall x,y \text{ homogeneous}. $$

Similarly, there is a unique $\mathbb{Q}(v)$-linear map

$$ \bar{r}_i : \mathbf{U}^- \to \mathbf{U}^- $$

defined by

$$ \bar{r}_i(1)=0, \quad \bar{r}_i(\theta_j)=\delta_{ij}, \quad \bar{r}_i(xy)=v^{(i,-|y|)}\bar{r}_i(x)y+x\bar{r}_i(y), \quad \forall x,y \text{ homogeneous}. $$

We have

Lemma 3.1.1. *For any homogeneous element $x\in\mathbf{U}^-$,*

$$ E_i(x)=(v^{(i,\lambda)}\bar{r}_i(x)-v^{-(i,\lambda-|x|+i)}\,{}_i\bar{r}(x))/(v-v^{-1}). \tag{10} $$

Proof. Let $\widetilde{E}_i(x)$ denote the right hand side of the equation (10). Then by induction and using the definition of ${}_i\bar{r}$ and $\bar{r}_i$, we have

$$ [\widetilde{E}_i,F_j](x)=\delta_{ij}\frac{K_i-K_{-i}}{v-v^{-1}}(x), \quad \forall x\in\mathbf{U}^-. $$

From this, we have

$$ [E_i-\widetilde{E}_i,F_j]=0, \quad \forall i,j\in I. $$

Observe that $(E_i-\widetilde{E}_i)(1)=0$. By using the fact that the monomials in $\mathbf{U}^-$ span the space $\mathbf{U}^-$, one can show, by induction, that $E_i(x)-\widetilde{E}_i(x)=0$ for all $x\in\mathbf{U}^-$ by using the above identity. ∎

Given any two weights λ and λ' in $\mathbf{X}$, we shall compare the actions E_i's that define the $\mathbf{U}$-module structures on $\mathbf{U}^-$ as M_λ and $M_{\lambda+\lambda'}$, respectively. To avoid confusion, we write E_i^λ for the E_i action on $\mathbf{U}^-$ defining the Verma module M_λ structure. Let ξ_λ denote the unit 1 in $\mathbf{U}^-$ if it is regarded as the Verma module M_λ.

By applying (9) repeatedly, we get

$$ \begin{aligned} E_i^\lambda(\theta_{i_1}\cdots\theta_{i_m}\xi_\lambda) &= \theta_{i_1}E_i^\lambda\theta_{i_2}\cdots\theta_{i_m}\xi_\lambda+\delta_{ii_1}\left([(i,\lambda-\sum_{n=2}^m\alpha_{i_n})]\right)\theta_{i_2}\cdots\theta_{i_m}\xi_\lambda \\ &= \sum_{k=1}^m\delta_{ii_k}\left([(i,\lambda-\sum_{n=k+1}^m\alpha_{i_n})]\right)\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_{k+1}}\cdots\theta_{i_m}\xi_\lambda. \end{aligned} $$

In short, we have

$$ E_i^\lambda(\theta_{i_1}\cdots\theta_{i_m}\xi_\lambda)=\sum_{k=1}^r\delta_{ii_k}\left([(i,\lambda-\sum_{n=k+1}^m\alpha_{i_n})]\right)\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_{k+1}}\cdots\theta_{i_m}\xi_\lambda. \tag{11} $$

Observe that

$$ [a+b]=v^b[a]+v^{-a}[b]. $$

From this and (11), we have

$$
\begin{aligned}
&E_i^{\lambda+\lambda'}(\theta_{i_1}\cdots\theta_{i_m}\xi_{\lambda+\lambda'})\\
&= v^{(i,\lambda')}E_i^{\lambda}(\theta_{i_1}\cdots\theta_{i_m}\xi_\lambda) + \sum_{k=1}^{m}\delta_{ii_k}\left(v^{-(i,\lambda-\sum_{m=k+1}^{n}\alpha_{i_n})}[(i,\lambda')]\right)\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_{k+1}}\cdots\theta_{i_m}\xi_\lambda\\
&= v^{(i,\lambda')}E_i^{\lambda}(\theta_{i_1}\cdots\theta_{i_m}\xi_\lambda) + v^{-(i,\lambda)}[(i,\lambda')]\sum_{k=1}^{m}\delta_{ii_k}v^{(i,\sum_{n=k+1}^{m}\alpha_{i_n})}\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_{k+1}}\cdots\theta_{i_m}\xi_\lambda
\end{aligned}
$$

Recall from [L93], there is a unique $\mathbb{Q}(v)$-linear map

$$r_i : \mathbf{U}^- \to \mathbf{U}^-$$

defined by

$$r_i(1) = 0, \quad r_i(\theta_j) = \delta_{ij} \quad \text{and} \quad r_i(xy) = v^{(i,|y|)}r_i(x)y + xr_i(y).$$

From the definition of r_i, we can prove by induction that

$$r_i(\theta_{i_1}\cdots\theta_{i_m}) = \sum_{k=1}^{m}\delta_{ii_k}v^{(i,\sum_{n=k+1}^{m}\alpha_{i_n})}\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_{k+1}}\cdots\theta_{i_m}.$$

So we have

$$E_i^{\lambda+\lambda'}(x\xi_{\lambda+\lambda'}) = v^{(i,\lambda')}E_i^{\lambda}(x\xi_\lambda) + v^{-(i,\lambda)}[(i,\lambda')]r_i(x).$$

The formula can be rewritten as

$$E_i^{\lambda}(x\xi_\lambda) = v^{-(i,\lambda')}E_i^{\lambda+\lambda'}(x\xi_{\lambda+\lambda'}) - v^{-(i,\lambda+\lambda')}[(i,\lambda')]r_i(x). \tag{12}$$

Note that

$$r_i(x) = v^{(i,|x|-\alpha_i)}\,{}_i\bar{r}(x), \quad \forall x \text{ homogeneous.}$$

By combining (12) and this, we have

Lemma 3.1.2. *$E_i^{\lambda}(x\xi_\lambda) = v^{-(i,\lambda')}E_i^{\lambda+\lambda'}(x\xi_{\lambda+\lambda'}) - v^{-(i,\lambda+\lambda'-|x|+\alpha_i)}[(i,\lambda')]\,{}_i\bar{r}(x)$, for any λ and λ' in $\mathbf{X}$ and any homogenous element $x \in \mathbf{U}^-$.*

This is the quantum analog of Lemma 10 in [GLS06].

3.2. **The module $\mathbf{K}'(\mathbf{d}^\bullet)$.** Let $\tilde{\Gamma}$ be the framed graph of Γ. This is a graph obtained from Γ by adding an extra copy of the vertex set, denoted by $I^+ = \{i^+ | i \in I\}$, and an edge joining i and i^+ for each $i \in I$.

Let $\mathbf{K}$ be the negative part $\mathbf{U}_{\tilde{\Gamma}}^-$ of the algebra $\mathbf{U}_{\tilde{\Gamma}}$ and we still write θ_j for the generators F_j in $\mathbf{K}$ when we regard it as a $\mathbf{U}_{\tilde{\Gamma}}$-module. By Section 3.1, $\mathbf{K}$ can be made into a Verma module of $\mathbf{U}_{\tilde{\Gamma}}$ of highest weight $\lambda \equiv 0$. Since Γ is a subgraph of $\tilde{\Gamma}$, we see that $\mathbf{U} = \mathbf{U}_\Gamma$ is a subalgebra of $\mathbf{U}_{\tilde{\Gamma}}$. By restriction to $\mathbf{U}$, we see immediately that $\mathbf{K}$ is a $\mathbf{U}$-module.

We would like to investigate the structure of the $\mathbf{U}$-module $\mathbf{K}$. For any $\alpha \in \mathbb{N}[\tilde{I}]$, where $\tilde{I}$ is the vertex set of $\tilde{\Gamma}$, we write $\alpha = \alpha_I + \alpha_{I^+}$ where α_I is the part supported on I while α_{I^+} the part on I^+. For any $\mathbf{d} = \sum_{i\in I} d_i i \in \mathbb{N}[I]$, we set

$$\mathbf{K}^{\mathbf{d}} = \{x \in \mathbf{K} \,|\, |x|_{I^+} = \sum_{i\in I} d_i i^+\}.$$

By induction, and using 3.1 (2), we see that $\mathbf{K}^{\mathbf{d}}$ is a $\mathbf{U}$-module. Moreover, we have

$$\mathbf{K} = \oplus_{\mathbf{d}\in\mathbb{Z}_{\geq 0}[I]}\mathbf{K}^{\mathbf{d}}. \tag{13}$$

For simplicity, we set

$$\theta^{\mathbf{d}} = \prod_{i\in I}\theta_{i^+}^{(d_i)}, \quad \forall \mathbf{d}\in\mathbb{N}[I].$$

Note that there is no ambiguity for the product since the generators θ_{i^+} commute with each other. For a sequence $\mathbf{d}^\bullet = (\mathbf{d}^1, \cdots, \mathbf{d}^m)$ of nonzero elements in $\mathbb{N}[I]$, we set $\mathbf{K}'(\mathbf{d}^\bullet)$ to be the subspace of $\mathbf{K}$ spanned by the elements of the form

$$x_1\theta^{\mathbf{d}^1}x_2\theta^{\mathbf{d}^2}x_3\cdots x_m\theta^{\mathbf{d}^m}x_{m+1}$$

where $x_1, \cdots, x_{m+1}\in\mathbf{U}^-$ and regarded as elements in $\mathbf{K}$ by the embedding $\mathbf{U}^-\to\mathbf{U}^-_{\tilde{\Gamma}} = \mathbf{K}$. Then, it is clear that $\mathbf{K}'(\mathbf{d}^\bullet)$ is a $\mathbf{U}$-module. In fact, we only need to check elements of the above form are stable under the E_i action. This can be proved by induction on $|x|_I$ and using again the recursive formula 3.1 (9). We set

$$|\mathbf{d}^\bullet| = \mathbf{d}^1 + \cdots + \mathbf{d}^m.$$

Now Section 3.1 (13) can be rewritten as

$$\mathbf{K} = \bigoplus_{\mathbf{d}\in\mathbb{N}[I]}\sum_{\mathbf{d}^\bullet:|\mathbf{d}^\bullet|=\mathbf{d}}\mathbf{K}'(\mathbf{d}^\bullet). \tag{14}$$

Thus to study $\mathbf{K}$, it reduces to study $\mathbf{K}'(\mathbf{d}^\bullet)$.

To each $\mathbf{d}\in\mathbb{N}[I]$, we fix a $\lambda\in X^+$ such that $(i,\lambda) = d_i$ for any $i\in I$. Define a linear map

$$\phi' : M_0 \to \mathbf{K}'(\mathbf{d})$$

by sending $x\xi_0\in M_0$ to $\theta^{\mathbf{d}}x$. One checks immediately that the following condition (15) holds for ϕ':

$$K_i\phi'(x) = v^{(i,\lambda)}\phi'(K_ix), \quad E_i\phi'(x) = \phi'(E_ix) \quad \forall x\in V_1, i\in I. \tag{15}$$

In other words, ϕ' is a $\mathbf{U}^{\geq 0}$-module homomorphism from $M_0\otimes\mathbb{Q}(v)_\lambda$ to $\mathbf{K}'(\mathbf{d})$. So, by Frobenius reciprocity (2), we have a unique $\mathbf{U}$-module homomorphism

$$\phi'_\lambda : M_0\otimes M_\lambda \to \mathbf{K}'(\mathbf{d}) \tag{16}$$

sending $x\xi_0\otimes\xi_\lambda$ to $\theta^{\mathbf{d}}x$, for any $x\in\mathbf{U}^-$. Moreover, we claim that

$$\phi'_\lambda(M_0\otimes T_\lambda) = 0. \tag{17}$$

This claim can be shown by the following two steps. The first step is the observation that, with $p = (i,\lambda)+1$,

$$\phi'_\lambda(y\xi_0\otimes\theta_i^{(m)}\xi_\lambda) = \left(\sum_{t=0}^{m}(-1)^t v^{-t(p-m)}\theta_i^{(m-t)}\theta^{\mathbf{d}}\theta_i^{(t)}\right)y, \quad y\in\mathbf{U}^-, m\in\mathbb{N}.$$

This identity can be proved by induction on m. If $m=p$, it simplifies to

$$\phi'_\lambda(y\xi_0\otimes\theta_i^{(m)}\xi_\lambda) = \left(\sum_{t=0}^{p}(-1)^t\theta_i^{(p-t)}\theta^{\mathbf{d}}\theta_i^{(t)}\right)y = \left(\sum_{t=0}^{p}(-1)^t\theta_i^{(p-t)}\theta_{i^+}^{(d_i)}\theta_i^{(t)}\right)\left(\prod_{j\neq i}\theta_{j^+}^{(d_j)}\right)y.$$

The term $\sum_{t=0}^{p}(-1)^t\theta_i^{(p-t)}\theta_{i^+}^{(d_i)}\theta_i^{(t)}$ is equal to zero because it is exactly the higher order quantum Serre relation $f_{i,j;n,m,e}$ with $i=i$, $j=i^+$ and $m=n+1=p$ in the algebra $\mathbf{f}$ attached to the graph $\tilde{\Gamma}$ in [L93, 7.1.1]. So we have

$$\phi'_\lambda(y\xi_0\otimes\theta_i^{(d_i+1)}\xi_\lambda)=0,\quad \forall y\in\mathbf{U}^-.$$

The second step is to observe that

$$\phi'_\lambda(y\xi_0\otimes x\theta_i^{(d_i+1)}\xi_\lambda)=0,\quad \forall x,y\in\mathbf{U}^-,$$

which can be proved by induction on the degree of x. The claim follows.

Now, from (16) and (17), we see that ϕ'_λ factors through the module $M_0\otimes V_\lambda$. In particular, we have a surjective homomorphism of $\mathbf{U}$ modules

$$\psi'_\lambda: M_0\otimes V_\lambda\twoheadrightarrow\mathbf{K}'(\mathbf{d}).\tag{18}$$

Let $\mathbf{T}'(\mathbf{d})$ be the submodule of $\mathbf{K}'(\mathbf{d})$ generated by the elements $x_1\theta^{\mathbf{d}}x_2$ such that x_2 is homogeneous and of degree $\neq 0$. In other words, $\mathbf{T}'(\mathbf{d})=\sum_{i\in I}y_i\theta^{\mathbf{d}}x_i\theta_i$, where $x_i,y_i\in\mathbf{U}^-$. Then the restriction of ψ'_λ to the submodule $T_0\otimes V_\lambda$ has image $\mathbf{T}'(\mathbf{d})$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & T_0\otimes V_\lambda \arrow[r]\arrow[d] & M_0\otimes V_\lambda \arrow[r]\arrow[d, "\psi'_\lambda"'] & V_0\otimes V_\lambda=V_\lambda \arrow[r]\arrow[d, "\bar\psi'_\lambda"'] & 0\\
0 \arrow[r] & \mathbf{T}'(\mathbf{d}) \arrow[r] & \mathbf{K}'(\mathbf{d}) \arrow[r] & \mathbf{V}'(\mathbf{d}) \arrow[r] & 0,
\end{tikzcd}$$

where we set

$$\mathbf{V}'(\mathbf{d})=\mathbf{K}'(\mathbf{d})/\mathbf{T}'(\mathbf{d}).$$

From the fact that ψ'_λ is surjective, we see that $\bar\psi'_\lambda$ is surjective:

$$\bar\psi'_\lambda: V_\lambda\twoheadrightarrow\mathbf{V}'(\mathbf{d}).\tag{19}$$

Since V_λ is simple and the generator $\theta^{\mathbf{d}}$ is not in $\mathbf{T}'(\mathbf{d})$, we see that $\bar\psi'_\lambda$ is an isomorphism.

We now generalize (18) and (19) to arbitrary cases.

For $\mathbf{d}^\bullet=(\mathbf{d}^1,\cdots,\mathbf{d}^N)$, we set $\mathbf{d}^{\bullet-1}=(\mathbf{d}^2,\cdots\mathbf{d}^N)$. Let λ^l be the chosen element in $\mathbb{N}[I]$ such that $(i,\lambda^l)=d_i^l$ for all $l=1,\cdots,N$. Let $\lambda^\bullet=(\lambda^N,\cdots,\lambda^1)$. Note that $\lambda^\bullet$ is in the reverse order of $\mathbf{d}^\bullet$. We set

$$V_{\lambda^\bullet}=V_{\lambda^N}\otimes\cdots\otimes V_{\lambda^1}.$$

Define a linear map

$$\phi'_{\lambda^\bullet}:\mathbf{K}'(\mathbf{d}^{\bullet-1})\to\mathbf{K}'(\mathbf{d}^\bullet)$$

by $\phi'_{\lambda^\bullet}(x)=\theta^{\mathbf{d}^1}x$, for any $x\in\mathbf{K}'(\mathbf{d}^{\bullet-1})$. One checks again that the condition (15) holds. So we have a unique $\mathbf{U}$-module homomorphism

$$\phi'_{\lambda^\bullet}:\mathbf{K}'(\mathbf{d}^{\bullet-1})\otimes M_{\lambda^1}\to\mathbf{K}'(\mathbf{d}^\bullet).$$

sending $x\otimes\xi_{\mathbf{d}^1}$ to $\theta^{\mathbf{d}^1}x$. Moreover, we may again prove that $\phi'(\mathbf{K}'(\mathbf{d}^{\bullet-1})\otimes T_{\lambda^1})=0$ in exactly the same manner as the proof of (17). So $\phi'_{\lambda^\bullet}$ factors through the module $\mathbf{K}'(\mathbf{d}^{\bullet-1})\otimes V_{\lambda^1}$. In particular, we have a surjective algebra homomorphism

$$\psi'_{\lambda^\bullet}:\mathbf{K}'(\mathbf{d}^{\bullet-1})\otimes V_{\lambda^1}\to\mathbf{K}'(\mathbf{d}^\bullet).$$

Let $\mathbf{T}'(\mathbf{d}^\bullet)$ be the submodule of $\mathbf{K}'(\mathbf{d}^\bullet)$ spanned by the elements $x_1\theta^{\mathbf{d}^1}x_2\theta^{\mathbf{d}^2}x_3\cdots x_N\theta^{\mathbf{d}^N}x_{N+1}$ such that is x_{N+1} homogeneous and of degree >0. Let

$$\mathbf{V}'(\mathbf{d}^\bullet)=\mathbf{K}'(\mathbf{d}^\bullet)/\mathbf{T}'(\mathbf{d}^\bullet),$$

be the quotient module. By induction, the homomorphism $V_{\lambda^{\bullet-1}} \twoheadrightarrow \mathbf{K}'(\mathbf{d}^{\bullet-1})/\mathbf{T}'(\mathbf{d}^{\bullet-1})$ is surjective. Moreover, $\psi'_{\lambda^\bullet}(\mathbf{T}'(\mathbf{d}^{\bullet-1})\otimes V_{\lambda^1})=\mathbf{T}'(\mathbf{d}^\bullet)$. So the morphism $\psi'_{\lambda^\bullet}$ induces a surjective homomorphism of $\mathbf{U}$ modules:

$$\bar{\psi}'_{\lambda^\bullet}:V_{\lambda^\bullet}\twoheadrightarrow \mathbf{V}'(\mathbf{d}^\bullet).$$

Summing up, we have the following proposition.

Proposition 3.2.1. *There is a unique surjective* $\mathbf{U}$*-module homomorphism*

$$\psi'_{\lambda^\bullet}:M_0\otimes V_{\lambda^\bullet}\twoheadrightarrow \mathbf{K}'(\mathbf{d}^\bullet),$$

sending $x\xi_0\otimes\xi_{\lambda^N}\otimes\cdots\otimes\xi_{\lambda^1}$ *to* $\theta^{\mathbf{d}^1}\theta^{\mathbf{d}^2}\cdots\theta^{\mathbf{d}^N}x$ *for any* $x\in\mathbf{U}^-$*. Moreover, it induces a surjective homomorphism of* $\mathbf{U}$*-modules* $\bar{\psi}'_{\lambda^\bullet}:V_{\lambda^\bullet}\twoheadrightarrow\mathbf{V}'(\mathbf{d}^\bullet)$*.*

3.3. **The module $\mathbf{K}(\mathbf{d}^\bullet)$.** Let $\mathbf{K}(\mathbf{d}^\bullet)$ be the subspace of $\mathbf{K}'(\mathbf{d}^\bullet)$ spanned by the elements of the form

$$x_1\theta^{\mathbf{d}^1}x_2\theta^{\mathbf{d}^2}x_3\cdots x_N\theta^{\mathbf{d}^N}$$

for any $x_1,\cdots,x_N\in\mathbf{U}^-$.

Note that by taking $\mathbf{d}^N=0$, we have $\mathbf{K}(\mathbf{d}^\bullet)=\mathbf{K}'(\mathbf{d}^\bullet)$ with $\mathbf{d}^\bullet=(\mathbf{d}^1,\cdots,\mathbf{d}^{N-1})$.

By a similar argument as in Section 3.2, we have a unique surjective $\mathbf{U}$-homomorphism

$$\psi_{\lambda^\bullet}:M_{\lambda^N}\otimes V_{\lambda^{N-1}}\otimes\cdots\otimes V_{\lambda^1}\twoheadrightarrow\mathbf{K}(\mathbf{d}^\bullet), \tag{20}$$

sending $x\xi_{\lambda^N}\otimes\cdots\otimes\xi_{\lambda^1}$ to $\theta^{\mathbf{d}^1}\cdots x\theta^{\mathbf{d}^N}$ for any $x\in\mathbf{U}^-$.

Let $\mathbf{T}(\mathbf{d}^\bullet)$ be the submodule of $\mathbf{K}(\mathbf{d}^\bullet)$ spanned by the elements $x_1\theta^{\mathbf{d}^1}x_2\theta^{\mathbf{d}^2}x_3\cdots x_N\theta_i^{d_i+1}\theta^{\mathbf{d}^N}$ for any $i\in I$ and $x_1,\cdots,x_N\in\mathbf{U}^-$. Let

$$\mathbf{V}(\mathbf{d}^\bullet)=\mathbf{K}(\mathbf{d}^\bullet)/\mathbf{T}(\mathbf{d}^\bullet),$$

be the quotient module. Since $\psi_{\lambda^\bullet}(T_{\lambda^N}\otimes V_{\lambda^{N-1}}\otimes\cdots\otimes V_{\lambda^1})\subseteq\mathbf{T}(\mathbf{d}^\bullet)$, we see that $\psi_{\lambda^\bullet}$ factors through the following surjective morphism

$$\bar{\psi}_{\lambda^\bullet}:V_{\lambda^\bullet}\twoheadrightarrow\mathbf{V}(\mathbf{d}^\bullet).$$

Remark 3.3.1. (1). When $N=1$, the isomorphism $\psi_\lambda:M_\lambda\to\mathbf{K}(\mathbf{d}^\bullet)$ is obvious.

(2). When $N=2$, the morphism $\psi'_\lambda:M_0\otimes V_{\lambda^1}\to\mathbf{K}(\mathbf{d}^\bullet)$ is an isomorphism, which is proved in Theorem 7.6.1 geometrically by identifying $\mathbf{K}(\mathbf{d}^\bullet)$ with the module $\mathcal{K}_{D^\bullet}$. Since the map $x_1\theta^{\mathbf{d}^1}x_2\mapsto x_1\theta^{\mathbf{d}^1}x_2\theta^{\mathbf{d}^2}$ defines an isomorphism $M_0\otimes V_{\lambda^1}\to M_{\lambda^2}\otimes V_{\lambda^1}$, as vector spaces. So the morphism $\psi_{\lambda^\bullet}:M_{\lambda^2}\otimes V_{\lambda^1}\to\mathbf{K}(\mathbf{d}^\bullet)$ is also an isomorphism. However, $\psi'_{\lambda^\bullet}$ and $\psi'_{\lambda^\bullet}$ are Not isomorphic in general. Indeed, we have $\mathbf{K}(\mathbf{d}^\bullet)=\mathbf{K}(\mathbf{d})$ in $\mathfrak{sl}_2$ case. Details will be given in a forthcoming paper.

4. Tensor product varieties

4.1. $\mathbf{E}(V)$**.** Recall from Section 2.1 that $\Gamma = (I, H)$ is a graph. For any finite dimensional I-graded vector space V over $\mathbb{C}$, we define

$$\mathbf{E}(V) = \oplus_{h \in H} \mathrm{Hom}(V_{h'}, V_{h''})$$

to be the representation space of the graph Γ of dimension vector $\nu = \sum_{i \in I} \dim V_i i$.

An element $x \in \mathbf{E}(V)$ is called *nilpotent* if there exists an integer n such that for any sequence $h_1, \cdots, h_n$ of arrows such that $h_1'' = h_2'$, $\cdots$, $h_{n-1}'' = h_n'$, the composition $x_{h_n} x_{h_{n-1}} \cdots x_{h_1}$ is equal to 0.

For any element $x \in \mathbf{E}(V)$ and I-graded subspace $W \subseteq V$, we say that W is x-invariant if $x_h(W_{h'}) \subseteq W_{h''}$ for any $h \in H$. In general, for any I-graded subspace U in V, we write $\overline{U}^x$ for the smallest x-invariant I-graded subspace in V containing U, and $\underline{U}^x$ for the largest x-invariant I-graded subspace contained in U. Whenever there is no ambiguity, we write $\overline{U}$ and $\underline{U}$ for $\overline{U}^x$ and $\underline{U}^x$, respectively.

Suppose that W is an x-invariant I-graded subspace. Let $T = V/W$ and fix an isomorphism $V \tilde{=} W \oplus T$. Then by restricting x to W, we obtain an element $x^{WW} \in \mathbf{E}(W)$. By passage to quotient, we obtain an element $x^{TT} \in \mathbf{E}(T)$. The element x^{WT} is the collection of linear maps $x_h : T_{h'} \to W_{h''}$ for $h \in H$. In other words, the h-component x_h of x can be written in the following block form:

$$x_h := \begin{pmatrix} x_h^{WW} & x_h^{WT} \\ 0 & x_h^{TT}. \end{pmatrix} : W_{h'} \oplus T_{h'} \to W_{h''} \oplus T_{h''}. \tag{21}$$

Moreover, one can show that

$$x \text{ is nilpotent if and only if } x^{TT} \text{ and } x^{WW} \text{ are nilpotent.} \tag{22}$$

4.2. $\mathbf{E}(V, D)$**.** Recall that $\tilde{\Gamma}$ is the framed graph of Γ. We fix a finite dimensional I^+-graded space D. Thus, the space

$$\mathbf{E}(V, D) = \mathbf{E}(V) \oplus \mathrm{Hom}(D, V) \oplus \mathrm{Hom}(V, D), \tag{23}$$

where $\mathrm{Hom}(V, D) = \oplus_{i \in I} \mathrm{Hom}(V_i, D_i)$ and $\mathrm{Hom}(D, V)$ is defined similarly, is the representation space $\mathbf{E}(V \oplus D)$ of the framed graph $\tilde{\Gamma}$. We shall identify $\mathbf{E}(V)$ with $\mathbf{E}(V, 0)$.

Elements in $\mathbf{E}(V, D)$ will be represented by $X = (x, p, q)$ with $x \in \mathbf{E}(V)$, $p \in \mathrm{Hom}(D, V)$ and $q \in \mathrm{Hom}(V, D)$. We also write

$$X(i) = \bigoplus_{\substack{h \in \tilde{\Gamma} \\ h' = i}} X_h = q_i \oplus \bigoplus_{\substack{h \in \Gamma \\ h' = i}} x_h \quad \text{and} \quad (i)X = \sum_{\substack{h \in \tilde{\Gamma} \\ h'' = i}} \epsilon(h) X_h = -p_i + \sum_{\substack{h \in \Gamma \\ h'' = i}} \epsilon(h) x_h. \tag{24}$$

The group $\mathbf{G}_V := \prod_{i \in I} \mathrm{GL}(V_i)$ acts on $\mathbf{E}(V, D)$ by conjugation, i.e.,

$$g.X = X^g, \quad X_h^g = \begin{cases} g_{h''} x_h g_{h'}^{-1}, & \text{if } h', h'' \in I, \\ q_i g_i^{-1}, & \text{if } h'' \notin I, \\ g_i p_i, & \text{if } h' \notin I, \end{cases} \qquad \forall g \in \mathbf{G}_V, X \in \mathbf{E}(V, D).$$

4.3. **Lusztig's and Nakajima's quiver varieties.** Let $\epsilon : H \to \{\pm 1\}$ be a map satisfying $\epsilon(h) + \epsilon(\bar{h}) = 0$ for any $h \in H$. The variety $\mathbf{\Lambda}_{V,D}$ is the subvariety of $\mathbf{E}(V, D)$ consisting of all elements (x, p, q) such that

$$\sum_{h \in H : h'' = i} \epsilon(h) x_h x_{\bar{h}} - p_i q_i = 0, \quad \forall i \in I.$$

Let $\mathbf{\Lambda}_{V \oplus D} \subseteq \mathbf{\Lambda}_{V,D}$ be Lusztig's Lagrangian nilpotent variety ([L90b], [L91]) attached to the graph $\tilde{\Gamma}$. More precisely, this is a closed subvariety of $\mathbf{\Lambda}_{V,D}$ determined by the conditions that X is nilpotent and $q_i p_i = 0$ for any $i \in I$. We write $\mathbf{\Lambda}_V$ for $\mathbf{\Lambda}_{V \oplus 0}$, which is a closed subvariety in $\mathbf{E}(V)$. Let

$$\mathbf{L}_{V,D} = \mathbf{\Lambda}_V \times \mathrm{Hom}(V, D),$$

and $\mathbf{L}^s_{V,D}$ be the open subvariety consisting of all elements X such that $X(i)$ (see (24)) are injective for any $i \in I$. It is well-known that $\mathbf{L}_{V,D}$ and $\mathbf{L}^s_{V,D}$ are pure dimensional, i.e. all irreducible components have the same dimension:

$$\dim \mathbf{L}_{V,D} = \dim \mathbf{L}^s_{V,D} = \frac{1}{2} \dim \mathbf{E}(V, D). \tag{25}$$

Note that $\mathbf{G}_V$ acts freely on $\mathbf{L}^s_{V,D}$ and its quotient $\mathfrak{L}_{V,D} = \mathbf{G}_V \backslash \mathbf{L}^s_{V,D}$ is the Lagrangian fiber of Nakajima's quiver variety ([N94]).

4.4. **Tensor product varieties $\mathbf{\Pi}_{V,D^\bullet}$.** We fix a decomposition

$$D = D^2 \oplus D^1, \quad \text{with} \quad \dim D^2 = d^2 \quad \text{and} \quad \dim D^1 = d^1,$$

and a flag $D^\bullet = (\check{D}^0 \supseteq \check{D}^1 \supseteq \check{D}^2)$ where $D = \check{D}^0$, $\check{D}^1 = D^2$ and $\check{D}^2 = 0$.

Let $\mathbf{\Pi}_{V,D^\bullet}$ be the closed subvariety of $\mathbf{\Lambda}_{V,D}$ consisting of all nilpotent elements $X = (x, p, q)$ such that

$$\overline{p(D)} \subseteq \underline{q^{-1}(D^2)} \quad \text{and} \quad p(D^2) = 0, \tag{26}$$

where the notations are defined in Section 4.1. (When $D^2 = 0$, the geometry of $\mathbf{\Pi}_{V,D^\bullet}$ corresponds to the module $\mathbf{K}'(\mathbf{d})$ in Section 3.2.)

Let $\mathbf{\Pi}_{V,D^\bullet;\nu^2}$ be the locally closed subvariety of $\mathbf{\Pi}_{V,D^\bullet}$ consisting of all elements such that $\dim \underline{q^{-1}(D^2)} = \nu^2$.

Fix an I-graded subspace V^2 of dimension ν^2 in V. Let $\mathbf{\Pi}_{V,D^\bullet;V^2}$ be the closed subvariety of $\mathbf{\Pi}_{V,D^\bullet;\nu^2}$ consisting of all elements such that $\underline{q^{-1}(D^2)} = V^2$.

Let $V^1 = V/V^2$ and fix a decomposition $V = V^2 \oplus V^1$. Then $D^2 \oplus V^2$ is X-invariant for any element $X \in \mathbf{\Pi}_{V,D^\bullet;V^2}$. Thus to each element $X = (x, p, q) \in \mathbf{\Pi}_{V,D^\bullet;V^2}$, attached the elements

$$X^1 = (x^{V^1V^1}, p^{V^1D^1}, q^{D^1V^1}) \in \mathbf{\Lambda}_{V^1,D^1}, \quad X^2 = (x^{V^2V^2}, p^{V^2D^2}, q^{D^2V^2}) \in \mathbf{\Lambda}_{V^2,D^2}, \tag{27}$$

and $(x^{V^2V^1}, p^{V^2D^1}, q^{D^2V^1})$ just as in (21). Since X is nilpotent, so are X^1 and X^2 by (22). Moreover, by the condition (26), we see that

$$p^{V^1D^1} = 0, \quad p^{V^2D^2} = 0 \quad \text{and} \quad \ker X^1(i) = 0, \quad \forall i \in I.$$

From the above analysis, the assignment $X \mapsto (X^1, X^2)$ defines a morphism

$$\rho : \mathbf{\Pi}_{V,D^\bullet;V^2} \to \mathbf{L}^s_{V^1,D^1} \times \mathbf{L}_{V^2,D^2}. \tag{28}$$

Lemma 4.4.1. *The morphism ρ is a vector bundle of fiber dimension*

$$\sum_{h\in H}\nu^1_{h'}\nu^2_{h''}+\sum_{i\in I}d^1_i\nu^2_i+\sum_{i\in I}\nu^1_i d^2_i-\sum_{i\in I}\nu^1_i\nu^2_i. \tag{29}$$

Proof. The proof is the same as the proof of Proposition 2.7 in [M03]. By keeping in mind the result (22), the fibre of ρ under a fixed point (X^1,X^2) is isomorphic to the vector subspace in $\oplus_{h\in H}\mathrm{Hom}(V^1_{h'},V^2_{h''})\oplus\mathrm{Hom}(D^1,V^2)\oplus\mathrm{Hom}(V^1,D^2)$ consisting of all elements $(x^{V^2V^1},p^{V^2D^1},q^{D^2V^1})$ such that

$$\sum_{h\in H:h''=i}\epsilon(h)(x_h^{V^2V^2}x_{\bar h}^{V^2V^1}+x_h^{V^2V^1}x_{\bar h}^{V^1V^1})-p_i^{V^2D^1}q_i^{D^1V^1}=0,\quad \forall i\in I. \tag{30}$$

So the fibre $\rho^{-1}(X^1,X^2)$ is isomorphic to the kernel of the following linear map

$$f:\oplus_{h\in H}\mathrm{Hom}(V^1_{h'},V^2_{h''})\oplus\mathrm{Hom}(D^1,V^2)\oplus\mathrm{Hom}(V^1,D^2)\to\mathrm{Hom}(V^1,V^2)$$

given by

$$(x^{V^2V^1},p^{V^2D^1},q^{D^2V^1})\mapsto\left(\sum_{h\in H:h''=i}\epsilon(h)(x_h^{V^2V^2}x_{\bar h}^{V^2V^1}+x_h^{V^2V^1}x_{\bar h}^{V^1V^1})-p_i^{V^2D^1}q_i^{D^1V^1}|i\in I\right).$$

The condition that X^1 is in $\mathbf{L}^s_{V^1,D^1}$ implies that f is surjective.

Indeed, let us consider the trace map

$$\mathrm{tr}:\mathrm{Hom}(V^1,V^2)\times\mathrm{Hom}(V^2,V^1)\to\mathbb{C},\quad (p^{V^2V^1},p^{V^1V^2})\mapsto\sum_{i\in I}\mathrm{tr}(p_i^{V^2V^1}p_i^{V^1V^2}),$$

where $\mathrm{tr}(p_i^{V^2V^1}p_i^{V^1V^2})$ is the trace of the endomorphism. Let $f^\perp$ be the perpendicular space of $\mathrm{im}(f)$ with respect to the trace map tr. Given any $q^{V^1V^2}$ in $f^\perp$, we have

$$\sum_{h\in H:h''=i}\epsilon(h)(\mathrm{tr}(x_h^{V^2V^2}x_{\bar h}^{V^2V^1}q_i^{V^1V^2})+\mathrm{tr}(x_h^{V^2V^1}x_{\bar h}^{V^1V^1}q_i^{V^1V^2}))-\mathrm{tr}(p_i^{V^2D^1}q_i^{D^1V^1}q_i^{V^1V^2})=0,$$

for any triple $(x^{V^2V^1},p^{V^2D^1},q^{D^2V^1})$ and $i\in I$. This implies that

$$\mathrm{tr}(x_h^{V^2V^2}x_{\bar h}^{V^2V^1}q_i^{V^1V^2})=0,\quad \mathrm{tr}(x_h^{V^2V^1}x_{\bar h}^{V^1V^1}q_i^{V^1V^2})=0,\quad \mathrm{tr}(p_i^{V^2D^1}q_i^{D^1V^1}q_i^{V^1V^2})=0,$$

for any triple $(x^{V^2V^1},p^{V^2D^1},q^{D^2V^1})$, any $i\in I$ and $h\in H$ such that $h''=i$. Since $x^{V^2V^1}$ and $p^{V^2D^1}$ are arbitrary, the last two equations imply that

$$x_{\bar h}^{V^1V^1}q_i^{V^1V^2}=0\quad\text{and}\quad q_i^{D^1V^1}q_i^{V^1V^2}=0,\quad \forall i\in I,h\in H \text{ such that } h''=i.$$

This is to say that $\mathrm{im}(q_i^{V^1V^2})\subseteq\ker X^1(i)$ for any $i\in I$. But the fact that $X^1\in\mathbf{L}^s_{V,D}$ implies that $\mathrm{im}(q_i^{V^1V^2})=0$ for any $i\in I$. In other words, $f^\perp=\{0\}$. Since the trace map tr is non degenerate, f is surjective.

It is obvious that the dimension of the kernel of f is exactly (29). The lemma follows. □

Given any variety $\mathbf{\Pi}$, we denote by $\mathrm{Irr}\,\mathbf{\Pi}$ the set of all irreducible components of $\mathbf{\Pi}$.

Proposition 4.4.2. *The following statements hold.*

1. $\mathbf{\Pi}_{V,D^\bullet;V^2}$ *has pure dimension of dimension* $\frac{1}{2}\dim\mathbf{E}(V,D)-\sum_{i\in I}\nu^1_i\nu^2_i$. *Moreover,* $\#\mathrm{Irr}\,\mathbf{\Pi}_{V,D^\bullet;V^2}=\#\mathrm{Irr}\,(\mathbf{L}^s_{V^1,D^1}\times\mathbf{L}_{V^2,D^2})$.
2. $\mathbf{\Pi}_{V,D^\bullet;\nu^2}$ *has pure dimension of dimension* $\frac{1}{2}\dim\mathbf{E}(V,D)$.

(3) $\mathbf{\Pi}_{V,D^\bullet}$ *has pure dimension of dimension* $\frac{1}{2}\dim \mathbf{E}(V,D)$.
(4) $\#\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet} = \sum_{V^1,V^2} \#\mathrm{Irr}\,(\mathbf{L}^s_{V^1,D^1} \times \mathbf{L}_{V^2,D^2})$, *where the sum runs over the representatives* (V^1,V^2) *of* (ν^1,ν^2) *such that* $\nu^1+\nu^2=\nu$.

Proof. The first statement follows from the fact that ρ is a vector bundle and that $\mathbf{L}_{V,D}$ and $\mathbf{L}^s_{V,D}$ are of pure dimension. The dimension of $\mathbf{\Pi}_{V,D^\bullet;V^2}$ can be computed by using (25) and (29).

Let $\mathrm{Gr}(\nu^2,V) = \times_{i\in I}\mathrm{Gr}(\nu^2_i,V_i)$ be the product of the Grassmannian of ν^2_i-dimensional vector subspaces in V_i for $i\in I$. The assignment $X \mapsto \underline{q^{-1}(D^2)}$ defines a surjective morphism $\mathbf{\Pi}_{V,D^\bullet;\nu^2} \to \mathrm{Gr}(\nu^2,V)$. It is clear that the fibre of the morphism at V^2 is $\mathbf{\Pi}_{V,D^\bullet;V^2}$ and the dimension of $\mathrm{Gr}(\nu^2,V)$ is $\sum_{i\in I}\nu^1_i\nu^2_i$. The second statement then follows.

Since $\mathbf{\Pi}_{V,D^\bullet} = \sqcup_{\nu^2}\mathbf{\Pi}_{V,D^\bullet;\nu^2}$, the third statement follows from the second one. So is the forth statement. The proposition is proved. □

4.5. **Finer structure on $\mathbf{\Pi}_{V,D^\bullet}$.** For any $i\in I$ and $n\in\mathbb{N}$, let

$$_{i,n}\mathbf{\Pi}_{V,D^\bullet} = \{X \in \mathbf{\Pi}_{V,D^\bullet} \,|\, \dim V_i/(i)X = n\},$$

where $(i)X$ is defined in (24). Suppose that the vector subspace W in V has dimension $\nu - ni$. Denote by $\mathrm{IHom}(V,W)$ the subset of $\mathrm{Hom}(V,W)$ consisting of injective maps. Let $\mathbf{\Pi}$ be the subvariety of $_{i,n}\mathbf{\Pi}_{V,D^\bullet} \times {}_{i,0}\mathbf{\Pi}_{W,D^\bullet} \times \mathrm{IHom}(V,W)$ consisting of all triples (X,X^1,R) such that

$$R_{h''}X^1_h = X_hR_{h'}, \forall h\in\tilde{\Gamma},$$

where $R_{h'}$ and $R_{h''}$ are understood to be identity morphism if h' or h'' are in I^+. Then we have a diagram

$$_{i,n}\mathbf{\Pi}_{V,D^\bullet} \xleftarrow{\pi_1} \mathbf{\Pi} \xrightarrow{\pi_2} {}_{i,0}\mathbf{\Pi}_{W,D^\bullet}, \tag{31}$$

where π_1 and π_2 are the obvious projections. Moreover, π_2 factors through $_{i,0}\mathbf{\Pi}_{W,D^\bullet} \times \mathrm{IHom}(V,W)$ via the obvious projections $\pi_2' : (X,X^1,R) \mapsto (X^1,R)$ and $\pi_2'' : (X^1,R) \to X^1$.

Lemma 4.5.1. *We have*

(1) π_1 *is a principal* $\mathbf{G}_W$*-bundle.*
(2) π_2' *is a vector bundle of dimension* $\sum_{h\in H:h'=i} n\nu_{h''} + nd_i - n(\nu_i - n)$.
(3) π_2'' *is a trivial bundle of dimension* $\dim \mathbf{G}_W + n(\nu_i - n)$.

Proof. The first and third statements are clear. The second one follows from an argument similar to the proof of Lemma 4.4.1 and [L91, 12.5]. □

The following proposition follows from Lemma 4.5.1.

Proposition 4.5.2. *Suppose that* $\dim W + ni = \dim V$. *The assignment* $Y \mapsto \pi_2(\pi_1^{-1}(Y))$ *defines a bijection*

$$\tilde{e}_i^{\max} : \mathrm{Irr}\, {}_{i,n}\mathbf{\Pi}_{V,D^\bullet} \to \mathrm{Irr}\, {}_{i,0}\mathbf{\Pi}_{W,D^\bullet}. \tag{32}$$

The assignment $Y' \mapsto \pi_1(\pi_2^{-1}(Y'))$ *defines a bijection*

$$\tilde{f}_i^n : \mathrm{Irr}\, {}_{i,0}\mathbf{\Pi}_{W,D^\bullet} \to \mathrm{Irr}\, {}_{i,n}\mathbf{\Pi}_{V,D^\bullet}. \tag{33}$$

Moreover the maps $\tilde{e}_i^{\max}$ *and* $\tilde{f}_i^n$ *are inverse to each other.*

4.6. **Crystal structure on the $\mathbf{\Pi}_{V,D^\bullet}$'s.** For each $\nu \in \mathbb{N}[I]$, we fix an I-graded vector space $V(\nu)$ of dimension ν. Let

$$\mathrm{Irr}\,(D^\bullet) = \sqcup_{\nu\in\mathbb{N}[I]} \mathrm{Irr}\,\mathbf{\Pi}_{V(\nu),D^\bullet}.$$

For any $Y \in \mathbf{\Pi}_{V,D^\bullet}$ and $i \in I$, we define

$$\begin{aligned}
&\mathrm{wt}(Y) = \lambda - \nu \in \mathbf{X},\\
&\varepsilon_i(Y) = n, \quad \text{if } Y \cap {}_{i,n}\mathbf{\Pi}_{V,D^\bullet} \text{ is open dense in } Y,\\
&\varphi_i = \varepsilon_i(Y) + (i, \mathrm{wt}(Y)),
\end{aligned}$$

where $\lambda \in \mathbf{X}$ is a fixed element such that $\lambda(i) = \dim D_i$ for all $i \in I$, $\nu \in \mathbf{X}$ is via the imbedding $\mathbb{N}[I] \to \mathbf{X}$. We also define

$$\tilde{f}_i(Y) = \tilde{f}_i^{\varepsilon_i(Y)+1}\tilde{e}_i^{\max}(Y) \quad \text{and} \quad \tilde{e}_i(Y) = \begin{cases} \tilde{f}_i^{\varepsilon_i(Y)-1}\tilde{e}_i^{\max}(Y) & \text{if } \varepsilon_i(Y) > 0,\\ 0 & \text{otherwise.}\end{cases}$$

In this way, we have defined the following five maps on $\mathrm{Irr}\,(D^\bullet)$:

$$\mathrm{wt} : \mathrm{Irr}\,(D^\bullet) \to \mathbf{X},\ \varepsilon_i, \varphi_i : \mathrm{Irr}\,(D^\bullet) \to \mathbb{Z} \text{ and } \tilde{e}_i, \tilde{f}_i : \mathrm{Irr}\,(D^\bullet) \to \mathrm{Irr}\,(D^\bullet) \sqcup \{0\}, \quad \forall i \in I.$$

The following lemma follows from a straightforward checking.

Lemma 4.6.1. *The data* $(\mathrm{Irr}\,(D^\bullet), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)_{i\in I}$ *form a crystal defined in Section 2.2. Moreover, the crystal* $\mathrm{Irr}\,(D^\bullet)$ *is generated by the set of all elements Y such that $\varepsilon_i(Y) = 0$ for all $i \in I$.*

We set

$$\mathrm{Irr}\,(\mathbf{L}^s_D) = \sqcup_{\nu\in\mathbb{N}[I]}\mathrm{Irr}\,\mathbf{L}^s_{V(\nu),D} \quad \text{and} \quad \mathrm{Irr}\,(\mathbf{L}_D) = \sqcup_{\nu\in\mathbb{N}[I]}\mathrm{Irr}\,\mathbf{L}_{V(\nu),D}.$$

On $\mathrm{Irr}\,(\mathbf{L}^s_D)$ and $\mathrm{Irr}\,(\mathbf{L}_D)$, one may define crystal structures in exactly the same way as the crystal structure on $\mathrm{Irr}\,(D^\bullet)$. Moreover, the crystal structure on $\mathrm{Irr}\,(\mathbf{L}^s_D)$ is isomorphic to $B(\lambda)$, while $\mathrm{Irr}\,(\mathbf{L}_D)$ isomorphic to $B(\lambda,\infty)$. See [KS97] and [N94] for details.

Define a map

$$\psi : \mathrm{Irr}\,(\mathbf{L}^s_{D^1}) \otimes \mathrm{Irr}\,(\mathbf{L}_{D^2}) \to \mathrm{Irr}\,(D^\bullet) \tag{34}$$

as follows. For any $Y^1 \in \mathrm{Irr}\,\mathbf{L}^s_{V^1,D^1}$ and $Y^2 \in \mathrm{Irr}\,\mathbf{L}_{V^2,D^2}$, we define $\psi(Y^1\otimes Y^2)$ to be the irreducible component $Y \in \mathrm{Irr}\,(\mathbf{\Pi}_{V,D^\bullet})$ such that $Y \cap \mathbf{\Pi}_{V,D^\bullet;V^2} = \rho^{-1}(Y^1\times Y^2)$ where ρ is defined in (28). By Proposition 4.4.2 (4), the map ψ is a bijection. Moreover,

Theorem 4.6.2. $\mathrm{Irr}\,(D^\bullet)$ *is isomorphic to* $\mathrm{Irr}\,(\mathbf{L}^s_{D^1}) \otimes \mathrm{Irr}\,(\mathbf{L}_{D^2}) = B(\lambda^1)\otimes B(\lambda^2,\infty)$ *as crystals via ψ, where λ^1 and λ^2 are fixed elements such that $\lambda^a(i) = \dim D^a_i$ for any $i \in I$ and $a = 1, 2$.*

Proof. The proof is similar to the proof of Theorem 4.6 in [N01]. We have to show that ψ is compatible with the five maps on $\mathrm{Irr}\,(D^\bullet)$ and $\mathrm{Irr}\,(\mathbf{L}^s_{D^1}) \otimes \mathrm{Irr}\,(\mathbf{L}_{D^2})$, respectively. It is obvious that $\mathrm{wt}(\psi(Y^1\otimes Y^2)) = \mathrm{wt}(Y^1) + \mathrm{wt}(Y^2)$.

Choose a triple (X, X^1, X^2) such that $\rho(X) = (X^1, X^2)$ in (28). Let us fix a decomposition $V = V^2 \oplus V^1$. Then the canonical short exact sequence $0 \to V^2 \to V \to V^1 \to 0$ induces a complex

$$\ker(i)X^2/\mathrm{im}X^2(i) \to \ker(i)X/\mathrm{im}X(i) \xrightarrow{\tilde{b}} \ker(i)X^1/\mathrm{im}X^1(i). \tag{35}$$

The condition (26) gives rise to a linear map

$$c : \ker(i)X^1/\mathrm{im}X^1(i) \to V_i^2/\mathrm{im}(i)X^2,$$

and $\tilde{b}c = 0$. So $\tilde{b}$ factors through $\ker(c)$. Thus the complex (35) induces a complex

$$0 \to \ker(i)X^2/\mathrm{im}X^2(i) \to \ker(i)X/\mathrm{im}X(i) \to \ker(c) \to 0, \tag{36}$$

which is indeed a short exact sequence. From (36), one deduces the following lemma.

Lemma 4.6.3. *Assume that $\psi(Y) = Y^1 \otimes Y^2$ satisfies $\varepsilon_i(Y) = 0$, then $\varepsilon_i(Y^1) = 0$ and $\dim\ker(c) = \mathrm{wt}_i(Y^1) - \varepsilon_i(Y^2)$.*

Suppose again that $\varepsilon_i(Y) = 0$ and that $X \in Y$ is a generic point. Fix a vector space T of dimension ri and $\tilde{V} = V \oplus T$. We consider the variety

$$\mathbf{\Pi}_X = \{\tilde{X} \in \mathbf{\Pi}_{\tilde{V},D^\bullet} : \tilde{X}|_V = X\}.$$

Just like (30), we deduce that $\mathbf{\Pi}_X$ can be identified with the vector subspace in

$$\mathrm{Hom}(T, D_i) \oplus_{h\in H:h''=i} \mathrm{Hom}(T_i, V_{h'})$$

consisting of all vectors $(q_i^{DT}, x_{\bar{h}}^{VT})_{h\in H:h''=i}$ such that

$$\sum_{h\in H:h''=i} \epsilon_i(h) x_h x_{\bar{h}}^{VT} - p_i q_i^{DT} = 0 \tag{37}$$

From the equation (37), one can deduce that the projection $q_i^{D^1T} + \sum_{h\in H:h''=i} x_{\bar{h}}^{V^1T} : T \to D_i^1 \oplus \oplus_{h\in H:h''=i} V_{h'}^1$ of q^{DT} and $x_{\bar{h}}^{VT}$ to D_i^1 and $V_{h'}^1$, respectively, has image contained in $\ker(i)X^1$. Moreover, the composition of this map with the projection from $\ker(i)X^1$ to $\ker(i)X^1/\mathrm{im}X^1(i)$ and the map c vanishes. So it gives rise to a linear map $\phi : T \to \ker(c)$. The assignment $(q^{DT}, x_{\bar{h}}^{VT}) \mapsto \phi$ then defines a linear map

$$f : \mathbf{\Pi}_X \to \mathrm{Hom}(T, \ker(c)).$$

Moreover, the map f is surjective. In fact, for any $\phi \in \mathrm{Hom}(T, \ker(c))$, one may define an element in $\mathbf{\Pi}_X$ as follows. Fix a decomposition $\ker(i)X^1 = \mathrm{im}X^1(i) \oplus \ker(i)X^1/\mathrm{im}X^1(i)$. Let $p_i^{D^1T}$ be the composition of ϕ and $\ker(c) \to \ker(i)X^1/\mathrm{im}X^1(i) \to \ker(i)X^1 \to D_i^1$. Similar $x_{\bar{h}}^{V^1T}$ be the composition of ϕ and $\ker(c) \to \ker(i)X^1/\mathrm{im}X^1(i) \to \ker(i)X^1 \to V_{h'}^1$. Then the condition (37) determines uniquely an element $x_{\bar{h}}^{V^2T}$. Now choose an arbitrary element $p_i^{D^2T}$. The element $(q_i^{DT}, x_{\bar{h}}^{VT})$ in $\mathbf{\Pi}_X$ is thus obtained, moreover, the element gets sent to ϕ via f. Therefore f is surjective.

Let $\mathrm{Hom}(T, \ker(c))^1$ be the open dense subvariety of $\mathrm{Hom}(T, \ker(c))$ consisting of all elements of maximal rank. Since f is surjective, we have that the inverse image $\mathbf{\Pi}_X^1$ of $\mathrm{Hom}(T, \ker(c))^1$ under f is open dense in $\mathbf{\Pi}_X$. Let $\mathbf{\Pi}_X^0$ be the subvariety in $\mathbf{\Pi}_X^1$ such that the corresponding element $\phi \in \mathrm{Hom}(T, \ker(c))$ satisfying $\ker\phi = \ker(q_i^{D^1T} + \sum_{h\in H:h''=i} x_{\bar{h}}^{V^1T})$. Then from the above explicit construction of elements in $\mathbf{\Pi}_X$, for any given ϕ, we see that

(38) $\mathbf{\Pi}_X^0$ is open dense in $\mathbf{\Pi}_X$ and, moreover, $\underline{(\tilde{q})^{-1}(D^2)} = V^2 \oplus \ker\phi$ for any $\tilde{X} \in \mathbf{\Pi}_X^0$.

Here $\tilde{X} = (\tilde{x}, \tilde{p}, \tilde{q})$ and $V^2 = \underline{q^{-1}(D^2)}$ for the fixed element X. From (38) and using (31), one can show the following lemma.

Lemma 4.6.4. *Suppose that Y is an irreducible component in $\mathrm{Irr}\,(D^\bullet)$ such that $\psi(Y) = Y^1 \otimes Y^2$ and $\varepsilon_i(Y) = 0$. Then one has*

$$\tilde{f}_i^r(Y) = \begin{cases} \tilde{f}_i^r Y^1 \otimes Y^2 & \text{if } r \leq \mathrm{wt}_i(Y^1) - \varepsilon_i(Y^2), \\ \tilde{f}_i^{\mathrm{wt}_i(Y^1)-\varepsilon_i(Y^2)} Y^1 \otimes \tilde{f}_i^{r-\mathrm{wt}_i(Y^1)+\varepsilon_i(Y^2)} Y^2 & \text{if } r > \mathrm{wt}_i(Y^1) - \varepsilon_i(Y^2). \end{cases} \tag{39}$$

Now by using Lemmas 4.6.1, 4.6.3 and 4.6.4 and using the proof of Lemma 4.11 in [N01], one can show that ψ is compatible with the five maps. □

4.7. **Crystal structure on the $\mathbf{\Pi}^s_{V,D^\bullet}$'s.** Let

$$\mathbf{\Pi}^s_{V,D^\bullet} = \{X \in \mathbf{\Pi}_{V,D^\bullet} | X(i) \text{ are injective for all } i \in I\}.$$

This is an open variety in $\mathbf{\Pi}_{V,D^\bullet}$. Similarly, one can define the open subvariety $\mathbf{\Pi}^s_{V,D^\bullet;V^2}$ in $\mathbf{\Pi}_{V,D^\bullet;V^2}$. Then we have the following cartesian diagram

$$\begin{array}{ccc} \mathbf{\Pi}^s_{V,D^\bullet;V^2} & \xrightarrow{\rho^s} & \mathbf{L}^s_{V^1,D^1} \times \mathbf{L}^s_{V^2,D^2} \\ \downarrow & & \downarrow \\ \mathbf{\Pi}_{V,D^\bullet;V^2} & \xrightarrow{\rho} & \mathbf{L}^s_{V^1,D^1} \times \mathbf{L}_{V^2,D^2}, \end{array} \tag{40}$$

where the bottom row is (28). From this diagram, we have

Proposition 4.7.1. *Statements similar to (1), (2) and (3) in Proposition 4.4.2 hold for the varieties $\mathbf{\Pi}^s_{V,D^\bullet;V^2}$, $\mathbf{\Pi}^s_{V,D^\bullet;\nu^2}$ and $\mathbf{\Pi}^s_{V,D^\bullet}$. Moreover,*

$$\#\mathrm{Irr}\,\mathbf{\Pi}^s_{V,D^\bullet} = \sum_{\nu^1,\nu^2} \#\mathrm{Irr}\,(\mathbf{L}^s_{V^1(\nu^1),D^1} \times \mathbf{L}^s_{V^2(\nu^2),D^2}),$$

where the sum runs over all pairs (ν^1, ν^2) such that $\nu^1 + \nu^2 = \nu$.

Note that the $\mathbf{G}_V$-action on $\mathbf{\Pi}^s_{V,D^\bullet}$ (resp. $\mathbf{L}^s_{V,D}$) is free and its quotient $\tilde{\mathfrak{Z}}_{V,D^\bullet}$ (resp. $\mathfrak{L}_{V,D}$) is the tensor product variety defined by Nakajima [N01] and Malkin [M03] (resp. Nakajima [N94]).

Let

$$\mathrm{Irr}\,(D^\bullet)^s = \sqcup_{\nu\in\mathbb{N}[I]} \mathrm{Irr}\,(\mathbf{\Pi}^s_{V(\nu),D^\bullet})$$

The inclusions $\mathbf{\Pi}^s_{V,D^\bullet} \subseteq \mathbf{\Pi}_{V,D^\bullet}$ define a surjective map, via restriction,

$$\iota : \mathrm{Irr}\,(D^\bullet) \to \mathrm{Irr}\,(D^\bullet)^s.$$

Similar to the crystal structure on $\mathrm{Irr}\,(D^\bullet)$, one can define a crystal structure on $\mathrm{Irr}\,(D^\bullet)^s$ with the help from a diagram similar to (31). It is proved in Theorem 4.6 in [N01] that the crystal structure on $\mathrm{Irr}\,(D^\bullet)^s$ is isomorphic to the crystal on the tensor product $B(\lambda^1)\otimes B(\lambda^2)$.

Corollary 4.7.2. *The surjective map $\iota : \mathrm{Irr}\,(D^\bullet) \to \mathrm{Irr}\,(D^\bullet)^s$ is the crystal morphism $B(\lambda^1) \otimes B(\lambda^2, \infty) \to B(\lambda^1) \otimes B(\lambda^2)$.*

Note that the crystal structure on $\mathrm{Irr}\,(D^\bullet)^s$ has been studied in [M03], [N01] and [S02].

5. Induction and restriction

We shall recall Lusztig's induction and restriction functors in this section. We refer to [L90]-[L93] for further reading.

5.1. **Lusztig's diagrams.** We preserve the setting in Section 4.2. Let Ω be an orientation of the framed graph $\tilde{\Gamma}$. This is a subset of the edge set $\tilde{H}$ of $\tilde{\Gamma}$ such that $\Omega \cup \overline{\Omega} = \tilde{H}$ and $\Omega \cap \overline{\Omega} = \emptyset$.

Let $\mathbf{E}_\Omega(V, D)$ be the subspace of $\mathbf{E}(V, D)$ consisting of all elements X such that $X_h = 0$ for any $h \notin \Omega$.

Recall that we fix a decomposition $D = D^2 \oplus D^1$ and $V = W \oplus T$. Let

$$\mathbf{F} = \{X \in \mathbf{E}_\Omega(V, D) | D^2 \oplus W \text{ is } X\text{-invariant}\}.$$

Then we have the following diagram

$$\mathbf{E}_\Omega(T, D^1) \times \mathbf{E}_\Omega(W, D^2) \xleftarrow{\kappa} \mathbf{F} \xrightarrow{\iota} \mathbf{E}_\Omega(V, D), \tag{41}$$

where ι is the closed embedding and $\kappa(X) = (X^1, X^2)$ with X^1 and X^2 defined in a similar way as (27). Note that κ is a vector bundle of fiber dimension $\sum_{h\in\Omega} \dim \tilde{T}_{h'} \dim \tilde{W}_{h''}$ where $\tilde{T}_{h'} = T_{h'}$ if $h' \in I$ and $\tilde{T}_{h'} = D^1_{h'}$ if $h' \in I^+$.

Let $\mathbf{P}_V$ be the stabilizer of W in $\mathbf{G}_V$ and $\mathbf{R}_V$ its unipotent radical. Then $\mathbf{P}_V/\mathbf{R}_V \simeq \mathbf{G}_T \times \mathbf{G}_W$. Consider the following diagram

$$\mathbf{E}_\Omega(T, D^1) \times \mathbf{E}_\Omega(W, D^2) \xleftarrow{\pi_1} \mathbf{G}_V \times_{\mathbf{R}_V} \mathbf{F} \xrightarrow{\pi_2} \mathbf{G}_V \times_{\mathbf{P}_V} \mathbf{F} \xrightarrow{\pi_3} \mathbf{E}_\Omega(V, D), \tag{42}$$

where $\pi_1(g, X) = \kappa(X)$, $\pi_2(g, X) = (g, X)$ and $\pi_3(g, X) = g.X$. It is well-known that π_3 is proper, π_2 is a $\mathbf{G}_T \times \mathbf{G}_W$-principal bundle and π_1 is a smooth morphism of connected fiber with fiber dimension $f_1 = \dim \mathbf{G}_V/\mathbf{R}_V + \dim \kappa^{-1}(X^1, X^2)$.

5.2. **Notations in derived categories.** We recall some notations on derived categories from [BBD82] and [L93]. Given any algebraic variety X over $\mathbb{C}$, denote by $\mathcal{D}(X)$ the bounded derived category of complexes of constructible sheaves on X. Denote by $\mathbb{C}_X$ the constant sheaf on X, regarded as the complex concentrated on degree zero. Let $[-]$ be the shift functor. Let $f : X \to Y$ be a morphism of varieties, denote by $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ and $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ the inverse image functor and the direct image functor with compact support, respectively. Let G be a connected algebraic group. Assume that G acts on X algebraically. Denote by $\mathcal{D}_G(X)$ the full subcategory of $\mathcal{D}(X)$ consisting of all G-equivariant complexes over X. Similarly, denote by $\mathcal{M}_G(X)$ the category of of all G-equivariant perverse sheaves on X. If G acts on X algebraically and f is a principal G-bundle, then f^* induces a functor, still denote by f^*, of equivalence between $\mathcal{M}(Y)[\dim G]$ and $\mathcal{M}_G(X)$. Its inverse functor is denoted by $f_\flat : \mathcal{M}_G(X) \to \mathcal{M}(Y)[\dim G]$.

5.3. **Induction and restriction functors.** From the diagram (41), we form the functor

$$\widetilde{\mathrm{Res}}^V_{T,W} = \kappa_! \iota^* : \mathcal{D}(\mathbf{E}_\Omega(V, D)) \to \mathcal{D}(\mathbf{E}_\Omega(T, D^1) \times \mathbf{E}_\Omega(W, D^2)). \tag{43}$$

From the diagram (42), we form the functor

$$\widetilde{\mathrm{Ind}}^V_{T,W} = \pi_{3!}\pi_{2\flat}\pi_1^* : \mathcal{D}_{\mathbf{G}_T\times\mathbf{G}_W}(\mathbf{E}_\Omega(T, D^1) \times \mathbf{E}_\Omega(W, D^2)) \to \mathcal{D}(\mathbf{E}_\Omega(V, D)). \tag{44}$$

The following shifted versions of the restriction and induction functors will also be used later.

$$\mathrm{Res}^V_{T,W} = \widetilde{\mathrm{Res}}^V_{T,W}[f_1 - f_2 - 2\dim \mathbf{G}_V/\mathbf{P}_V], \quad \mathrm{Ind}^V_{T,W} = \widetilde{\mathrm{Ind}}^V_{T,W}[f_1 - f_2], \tag{45}$$

where f_1 and f_2 are the fiber dimensions of the morphisms π_1 and π_2, respectively. For simplicity, we write

$$K_1 \cdot K_2 = \mathrm{Ind}^V_{T,W}(K_1 \boxtimes K_2).$$

5.4. **Special cases.** The following special cases of the functors $\mathrm{Res}^V_{T,W}$ and $\mathrm{Ind}^V_{T,W}$ will be used extensively later.

The first case is when $D^1 = 0$ and $\dim T = ni$. In this case, $\mathbf{E}_\Omega(T, D^1) = 0$. The shifted induction and restriction functors induce the following functors

$$ {}_i\mathcal{R}^{(n)} : \mathcal{D}(\mathbf{E}_\Omega(V, D)) \to \mathcal{D}(\mathbf{E}_\Omega(W, D)), \quad \mathcal{F}_i^{(n)} : \mathcal{D}(\mathbf{E}_\Omega(W, D)) \to \mathcal{D}(\mathbf{E}_\Omega(V, D)). \tag{46}$$

The second case is when $D^2 = 0$ and $\dim W = ni$. In this case, $\mathbf{E}_\Omega(W, D^2) = 0$. The shifted restriction functor induces the following functor

$$\mathcal{R}_i^{(n)} : \mathcal{D}(\mathbf{E}_\Omega(V, D)) \to \mathcal{D}(\mathbf{E}_\Omega(T, D)). \tag{47}$$

The third case is when $T = 0$. In this case, $\mathbf{E}_\Omega(T, D^1) = 0$ and $\pi_3 = \iota$ is a closed embedding. Thus, the functor $\mathrm{Ind}^V_{T,W}$ induces a fully faithful functor

$$L_{d^1} \cdot : \mathcal{D}(\mathbf{E}_\Omega(V, D^2)) \to \mathcal{D}(\mathbf{E}_\Omega(V, D)), \quad K_2 \mapsto L_{d^1} \cdot K_2. \tag{48}$$

The last case is when $W = 0$. In this case, $\mathbf{E}_\Omega(W, D^2) = 0$ and π_3 is the identity morphism. Thus the functor $\mathrm{Ind}^V_{T,W}$ induces a fully faithful functor

$$\cdot L_{d^2} : \mathcal{D}(\mathbf{E}_\Omega(T, D^1)) \to \mathcal{D}(\mathbf{E}_\Omega(V, D)), \quad K_1 \mapsto K_1 \cdot L_{d^2}. \tag{49}$$

6. Geometric study of M_λ

6.1. **Perverse sheaves on $\mathbf{E}_\Omega(V)$.** Let

$$\mathbf{E}_\Omega(V) = \mathbf{E}_\Omega(V, 0),$$

where $\mathbf{E}_\Omega(V, 0)$ is $\mathbf{E}_\Omega(V, D)$ in Section 5.1 for $D = 0$.

Let $\mathbf{i} = (i_1, \cdots, i_m)$ be a sequence of vertices in Γ and $\mathbf{a} = (a_1, \cdots, a_m)$ a sequence of non negative integers such that $a_1 i_1 + \cdots + a_m i_m = \nu$. A flag $F^\bullet = (V = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m = 0$ is of type $(\mathbf{i}, \mathbf{a})$ if $\dim F^l/F^{l+1} = a_{l+1} i_{l+1}$ for all $0 \leq l \leq m-1$. The variety $\widetilde{\mathbf{F}}_{\mathbf{i},\mathbf{a}}$ is the variety of all pairs $(x, F^\bullet)$, where $x \in \mathbf{E}_\Omega(V)$ and $F^\bullet$ is a flag of type $(\mathbf{i}, \mathbf{a})$, such that each subspace in $F^\bullet$ is x-invariant. Let

$$\pi_{\mathbf{i},\mathbf{a}} : \widetilde{\mathbf{F}}_{\mathbf{i},\mathbf{a}} \to \mathbf{E}_\Omega(V) \tag{50}$$

be the projection to the first component. We set

$$L_{(\mathbf{i},\mathbf{a})} = \pi_{\mathbf{i},\mathbf{a}!}(\mathbb{C}_{\widetilde{\mathbf{F}}_{\mathbf{i},\mathbf{a}}}[\dim \widetilde{\mathbf{F}}_{\mathbf{i},\mathbf{a}}]). \tag{51}$$

Since $\widetilde{\mathbf{F}}_{\mathbf{i},\mathbf{a}}$ is smooth and $\pi_{\mathbf{i},\mathbf{a}}$ is proper, the complex $L_{(\mathbf{i},\mathbf{a})}$ is semisimple by the decomposition theorem in [BBD82]. Let $\mathcal{Q}_V$ be the full 'semisimple' subcategory of $\mathcal{D}(\mathbf{E}_\Omega(V))$ whose simple objects are isomorphic to those appeared in $L_{(\mathbf{i},\mathbf{a})}$ for various pairs $(\mathbf{i}, \mathbf{a})$.

Let $\mathcal{N}_{V,i}$ be the full subcategory of $\mathcal{Q}_V$ whose simple objects are isomorphic to those appeared in $L_{(\mathbf{i},\mathbf{a})}$ for various pairs $(\mathbf{i}, \mathbf{a})$ such that the last term i_m of $\mathbf{i}$ is i and $a_m \geq d_i + 1$.

Let

$$\mathbf{E}_{\Omega, i, \geq d_i+1}(V) := \{x \in \mathbf{E}_\Omega(V) | \dim \ker x(i) \geq d_i + 1\},$$

where $x(i)$ is defined in (24).

Lemma 6.1.1. *Suppose that i is a source in Ω. Then any complex K in $\mathcal{Q}_V$ is in $\mathcal{N}_{V,i}$ if and only if* $\mathrm{Supp}(K) \subseteq \mathbf{E}_{\Omega,i,\geq d_i+1}(V)$.

This can be shown by the detailed analysis in [L93, 9.3].

Let $\mathcal{N}_V$ be the full subcategory of $\mathcal{Q}_V$ generated by $\mathcal{N}_{V,i}$ for all $i \in I$. Thus any simple object $K \in \mathcal{Q}_V$ is in $\mathcal{N}_V$ if and only if

$$\mathrm{Supp}(\Phi_\Omega^{\Omega_i} K) \subseteq \mathbf{E}_{\Omega_i,i,\geq d_i+1}(V), \quad \text{for some } i \text{ in } I. \tag{52}$$

where Ω_i is an orientation of $\tilde{\Gamma}$ with i a source and

$$\Phi_\Omega^{\Omega_i} : \mathcal{D}(\mathbf{E}_\Omega(V)) \to \mathcal{D}(\mathbf{E}_{\Omega_i}(V)),$$

is a fixed Fourier transform from $\mathcal{D}(\mathbf{E}_\Omega(V))$ to $\mathcal{D}(\mathbf{E}_{\Omega_i}(V))$ (see [L93], [L91], [KS90]).

6.2. **Perverse sheaves on $\mathbf{E}_\Omega(V, D)$.** Fix an order, say $(i_1^+, i_2^+, \cdots, i_N^+)$, of the set I^+. Let d denote a fixed pair of sequences $(i_1^+, \cdots, i_N^+)$ and $(d_1, \cdots, d_N)$. Let $(\mathbf{i}, \mathbf{a})\cdot d$ be the composition of the pair $(\mathbf{i}, \mathbf{a})$ and the pair d.

Then the complex $L_{(\mathbf{i},\mathbf{a})\cdot d}$ is well defined and semisimple over $\mathbf{E}_\Omega(V, D)$. Moreover, the complex $L_{(\mathbf{i},\mathbf{a})\cdot d}$ is independent of the choice of the order $(i_1^+, \cdots, i_N^+)$ of I^+. This is because $L_{(\mathbf{i},\mathbf{a})\cdot d}$ is equal to $L_{\mathbf{i},\mathbf{a}} \cdot L_d$ where $\cdot L_d$ is defined in Section 5.4. Observe that $\mathbf{E}_\Omega(0, d)$ is a single point. $L_d = \mathbb{C}_{\mathbf{E}_\Omega(0,d)}$, which is independent of the choice of the order on I^+.

Let $\mathcal{Q}_{V,D}$ be the full subcategory of $\mathcal{D}(\mathbf{E}_\Omega(V, D))$ defined with respect to the complexes $L_{(\mathbf{i},\mathbf{a})\cdot d}$ for various $(\mathbf{i}, \mathbf{a})$ in a similar way as $\mathcal{Q}_V$ to the complexes $L_{(\mathbf{i},\mathbf{a})}$.

Let $\pi : \mathbf{E}_\Omega(V, D) \to \mathbf{E}_\Omega(V)$ be the obvious projection. It then induces a functor, the shifted inverse image functor,

$$\pi^*[\nu d_\Omega] : \mathcal{D}(\mathbf{E}_\Omega(V)) \to \mathcal{D}(\mathbf{E}_\Omega(V, D)), \quad \text{where } \nu d_\Omega = \sum_{i \in I : i \to i^+ \in \Omega} \nu_i d_i.$$

Lemma 6.2.1. *We have $\cdot L_d = \pi^*[\nu d_\Omega]$; moreover, they are functors of equivalence from $\mathcal{Q}_V$ to $\mathcal{Q}_{V,D}$.*

In fact, the vector subspace $0 \oplus D$ is the only vector subspace in $V \oplus D$ of dimension $\sum_{i\in I} d_i i^+$ that is invariant under a fixed element X in $\mathbf{E}_\Omega(V, D)$. The isomorphism $\cdot L_d = \pi^*[\nu d_\Omega]$ follows from this and the definition of the multiplication "$\mathrm{Ind}_{T,W}^V$" in Section 5.1. They are equivalent because π^* is a fully faithful functor due to the fact that π is a trivial vector bundle of fiber dimension νd_Ω.

Let $\mathcal{N}_{V,D} = \pi^*[\nu d_\Omega](\mathcal{N}_V)$, i.e., the full subcategory of $\mathcal{Q}_{V,D}$ whose objects are of the form $\pi^*(K)$ with $K \in \mathcal{N}_V$. By Lemma 6.2.1, the condition (52) can be restated as follows.

Lemma 6.2.2. *Any simple object $K \in \mathcal{Q}_{V,D}$ is in $\mathcal{N}_{V,D}$ if and only if*

$$\mathrm{Supp}(\Phi_\Omega^{\Omega_i} K) \subseteq \mathbf{E}_{\Omega_i,i,\geq 1}(V, D), \quad \textit{for some } i \textit{ in } I, \tag{53}$$

where $\mathbf{E}_{\Omega,i,\geq 1}(V, D) = \{X \in \mathbf{E}_\Omega(V, D) | \dim \ker X(i) \geq 1\}$ and $X(i)$ is defined in (24).

Another way of stating (53) is

$$\mathrm{Supp}(\Phi_\Omega^{\Omega_i} K) \cap (\mathbf{E}_{\Omega_i}(V, D) \backslash \mathbf{E}_{\Omega_i,i,\geq 1}(V, D)) = \emptyset, \quad \text{for some } i \text{ in } I. \tag{54}$$

Let $\mathcal{V}_{V,D}$ (resp. $\mathcal{V}_V$) be the localization of $\mathcal{Q}_{V,D}$ with respect to the subcategory $\mathcal{N}_{V,D}$ (resp. $\mathcal{N}_V$). The above analysis produces the following commutative diagram of functors

$$\begin{array}{ccccc} \mathcal{N}_V & \xrightarrow{\iota} & \mathcal{Q}_V & \xrightarrow{Q} & \mathcal{V}_V \\ \downarrow & & \downarrow{\scriptstyle \pi^*[\nu d_\Omega]} & & \downarrow \\ \mathcal{N}_{V,D} & \xrightarrow{\iota} & \mathcal{Q}_{V,D} & \xrightarrow{Q} & \mathcal{V}_{V,D}, \end{array} \tag{55}$$

where the ι's are natural embedding, the Q's are localization functors, and the unexplained vertical maps are induced from $\pi^*[\nu d_\Omega]$.

Remark 6.2.3. (i) The condition (54) is exactly the stability condition used in [Zh08] for localizing the category $\mathcal{Q}_{V,D}$ to get the module V_λ for a chosen dominant weight λ such that $(i, \lambda) = d_i$ for any $i \in I$.

(ii) Note that the rows in the diagram (55) is exactly the categorical version of the short exact sequence (3).

(iv) As I.B. Frenkel pointed out, the sequence (3) is the first step of the BGG resolution of the module V_λ a q-analogue of the resolution in [BGG75]. It is very interesting to investigate the possibility of naturally lifting the BGG solution to the categorical level.

6.3. **Stability conditions for $\mathcal{Q}_{V,D}$.** In the section, we will use the notations in Section 4.3 freely. By [L91, 13], we have

$$\mathrm{SS}(K) \subseteq \mathbf{\Lambda}_{V\oplus D}, \quad \forall K \in \mathcal{Q}_{V,D}. \tag{56}$$

Moreover, by using [L91, Theorem 13.3],

$$\mathrm{SS}(L_{(\mathbf{i},\mathbf{a})\cdot d}) = \mathrm{SS}(L_{(\mathbf{i},\mathbf{a})} \cdot L_d) \subseteq \{p = 0\} \cap \mathbf{\Lambda}_{V\oplus D}. \tag{57}$$

If $K \in \mathcal{Q}_{V,D}$ is simple up to a shift, then K is a direct summand of the semisimple complex of the form $L_{(\mathbf{i},\mathbf{a})} \cdot L_d$. Thus, by (56) and (57),

$$\mathrm{SS}(K) \subseteq \mathbf{L}_{V,D}, \quad \forall K \in \mathcal{Q}_{V,D}.$$

Let $\mathcal{M}_{V,D}$ be the full subcategory of $\mathcal{Q}_{V,D}$ consisting of objects K such that

$$\mathrm{SS}(K) \cap \mathbf{L}^s_{V,D} = \emptyset. \tag{58}$$

Since the condition (58) does not involve any orientation of the graph Γ (or $\tilde{\Gamma}$), we call it a *global condition*, while (54) is called a *local condition*.

Proposition 6.3.1. *We have $\mathcal{N}_{V,D} = \mathcal{M}_{V,D}$.*

Proof. It is clear from (53) or (54) that if $K \in \mathcal{N}_{V,D}$, then $\mathrm{SS}(K) \cap \mathbf{L}^s_{V,D} = \emptyset$. So we have $\mathcal{N}_{V,D} \subseteq \mathcal{M}_{V,D}$.

By Theorem 6.2.2 in [KS97], for each simple perverse sheaf K in $\mathcal{Q}_{V,D}$, one can associate an irreducible component, say Y_K, such that

$$Y_K \subset \mathrm{SS}(K) \subset Y_K \cup \cup_{K' : \varepsilon_i(K') \geq \varepsilon_i(K)} Y_{K'},$$

for all $i \in I$. Moreover, if $K \neq K'$, then $Y_K \neq Y_{K'}$.

Note that the inequality is a strictly inequality in [KS97], which is a typo. See Remark 4.27 in [Sch09]. For a proof of this inequality, see [K07].

Thus the assignment $K \mapsto Y_K$ defines an injective map

$$\phi : S_1 \hookrightarrow S_2,$$

where S_1 is the set of isomorphism classes of simple perverse sheaves in $\mathcal{N}_{V,D}$ and S_2 is the set of irreducible components in $\mathbf{L}_{V,D}$ disjoint from $\mathbf{L}^s_{V,D}$. Observe that the sets S_1 and S_2 have the same number of elements, equal to $\dim T_{\lambda,\nu}$ due to [L93] and [L00a]. So ϕ is bijective. This implies that $\mathcal{N}_{V,D} = \mathcal{M}_{V,D}$. Otherwise, if $K \in \mathcal{M}_{V,D} \backslash \mathcal{N}_{V,D}$ is simple, then $Y_K \cap L^s_{V,D} = \emptyset$ by definition. Since ϕ is bijective, there is a $K' \in \mathcal{N}_{V,D}$ such that $Y_{K'} = Y_K$. This contradicts with the fact that $K \neq K'$ implies that $Y_K \neq Y_{K'}$. Proposition follows. $\square$

From Proposition 6.3.1, we have the following corollary.

Corollary 6.3.2. *Assume that $K \in \mathcal{Q}_{V,D}$. $Q(K) \neq 0$ if and only if $\mathrm{SS}(K) \cap \mathbf{L}^s_{V,D} \neq \emptyset$.*

Let $\mathcal{P}_{V,D}$ be the set of all isomorphism classes of simple perverse sheaves in $\mathcal{Q}_{V,D}$. Let $\mathcal{P}^s_{V,D}$ be the subset of $\mathcal{P}_{V,D}$ consisting of all elements not in $\mathcal{N}_{V,D}$. Then we have the following corollary.

Corollary 6.3.3. *For any $K \in \mathcal{P}_{V,D}$, the following statements are equivalent.*

(1) $K \in \mathcal{P}^s_{V,D}$. (2) $Q(K) \neq 0$. (3) $\mathrm{SS}(K) \cap \mathbf{L}^s_{V,D} \neq \emptyset$.

(4) $\mathrm{Supp}(\Phi^{\Omega_i}_{\Omega}(K)) \cap \mathbf{E}_{\Omega_i,i,0}(V,D)$ *is open dense in* $\mathrm{Supp}(\Phi^{\Omega_i}_{\Omega}(K))$, *where* $\mathbf{E}_{\Omega_i,i,0}(V,D) = \{X \in \mathbf{E}_{\Omega_i}(V,D) \mid \dim \ker X(i) = 0\}$ *for any* $i \in I$.

Remark 6.3.4. The results in the section are the combination of the work [L91] and [KS97]. Results in [N94, 11] are closely related to the results in this section.

7. Geometric study of tensor product $M_{\lambda^2} \otimes V_{\lambda^1}$

7.1. **Tensor product complexes.** In this section, we fix three elements λ, λ^1, λ^2 in $\mathbf{X}^+$ such that

$$\lambda^1 + \lambda^2 = \lambda.$$

This matches with the fixed decomposition $D = D^2 \oplus D^1$ in Section 4.4 by assuming that $(i,\lambda) = \dim D_i$ and $(i,\lambda^a) = \dim D^a_i$ for any $i \in I$ and $a = 1,2$. Let d^a be the dimension vector of D^a for $a = 1,2$.

In this section, we will consider the compositions

$$\underline{\mathbf{a}} := (\mathbf{i}^1, \mathbf{a}^1) \cdot d^1 \cdot (\mathbf{i}^2, \mathbf{a}^2) \cdot d^2$$

such that $\sum_{k=1}^{m^1} a^1_k i^1_k + \sum_{k=1}^{m^2} a^2_k i^2_k = \nu$ where ν is the dimension vector of V in the space $\mathbf{E}_\Omega(V,D)$. We can define the map

$$\pi^l_{\underline{\mathbf{a}}} : \widetilde{\mathbf{F}}_{\underline{\mathbf{a}}} \to \mathbf{E}_\Omega(V,D),$$

in exactly the same manner as the map $\pi_{\mathbf{i},\mathbf{a}}$ defined in (50).

Fix a partial flag $\check{D}^\bullet = (\check{D}^0 = D \supseteq \check{D}^1 \supseteq 0)$ such that $\dim \check{D}^0/\check{D}^1 = d^1$. In other words, we fix a subspace $\check{D}^1$ of D of dimension d^2. Let $\widetilde{\mathbf{E}}_{\underline{\mathbf{a}}}$ be the subvariety of $\widetilde{\mathbf{F}}_{\underline{\mathbf{a}}}$ consisting of all pairs whose flags incident to D is $\check{D}^\bullet$.

The restriction of $\pi^l_{\underline{\mathbf{a}}}$ to $\widetilde{\mathbf{E}}_{\underline{\mathbf{a}}}$ is denoted by

$$\pi_{\underline{\mathbf{a}}} : \widetilde{\mathbf{E}}_{\underline{\mathbf{a}}} \to \mathbf{E}_\Omega(V,D),$$

in exactly the same manner as the map $\pi_{(\mathbf{i},\mathbf{a})}$ defined in (50). Note that $\widetilde{\mathbf{E}}_{\underline{\mathbf{a}}}$ is again a smooth irreducible variety. So we have the following semisimple complex

$$L_{\underline{\mathbf{a}}} := \pi_{\underline{\mathbf{a}}!}(\mathbb{C}_{\widetilde{\mathbf{E}}_{\underline{\mathbf{a}}}})[\dim \widetilde{\mathbf{E}}_{\underline{\mathbf{a}}}]. \tag{59}$$

Note that when $\mathbf{a}^2 = 0$, $L_{\underline{\mathbf{a}}}$ is the same as $L_{(\mathbf{i},\mathbf{a})\cdot d}$.

Similar to $\mathcal{Q}_{V,D}$, let $\mathcal{Q}_{V,D^\bullet}$ be the full 'semisimple' subcategory of $\mathcal{D}(\mathbf{E}_\Omega(V,D))$ whose simple objects are isomorphic to those appeared in $L_{\underline{\mathbf{a}}}$ for various compositions $\underline{\mathbf{a}}$.

Similar to $\mathcal{N}_{V,D}$, we define $\mathcal{N}_{V,D^\bullet}$ to be the full subcategory of $\mathcal{Q}_{V,D^\bullet}$ generated by the simple objects K satisfying the local stability condition (53).

Similar to $\mathcal{V}_{V,D}$, we define $\mathcal{V}_{V,D^\bullet}$ to be the localization of the category $\mathcal{Q}_{V,D^\bullet}$ with respect to $\mathcal{N}_{V,D^\bullet}$. Altogether, we have the following exact sequence of categories

$$\mathcal{N}_{V,D^\bullet} \to \mathcal{Q}_{V,D^\bullet} \to \mathcal{V}_{V,D^\bullet}. \tag{60}$$

Since $L_{(\mathbf{i},\mathbf{a})\cdot d}$ is a special case of the complex $L_{\underline{\mathbf{a}}}$, we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{N}_{V,D} & \longrightarrow & \mathcal{Q}_{V,D} & \longrightarrow & \mathcal{V}_{V,D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N}_{V,D^\bullet} & \longrightarrow & \mathcal{Q}_{V,D^\bullet} & \longrightarrow & \mathcal{V}_{V,D^\bullet}, \end{array} \tag{61}$$

where the top level is from (55) and the vertical maps are inclusions. While the diagram (55) indicates that we go from non framed situation to framed situation. Here, this diagram (61) indicates that we go from the Verma module to the tensor product of a Verma module with a simple module.

7.2. **Structures on $\mathcal{P}_{V,D^\bullet}$.** Let ${}_i\Omega$ be an orientation of $\tilde{\Gamma}$ such that i is a sink, i.e., all arrows, h, adjacent to i has $h'' = i$. Let ${}_{i,n}\mathbf{E}_{{}_i\Omega}(V,D)$ be the locally closed subvariety of $\mathbf{E}_{{}_i\Omega}(V,D)$ consisting of all elements X such that $\dim V_i/(i)X = n$. Here $(i)X$ is defined in (24). We also set

$${}_{i,\geq n}\mathbf{E}_{{}_i\Omega}(V,D) = \sqcup\, {}_{i,n'}\mathbf{E}_{{}_i\Omega}(V,D),$$

where the union runs over all n' such that $n' \geq n$, which is a closed subvariety of $\mathbf{E}_{{}_i\Omega}(V,D)$.

Let $\mathcal{P}_{V,D^\bullet}$ be the set of isomorphic classes of simple perverse sheaves in $\mathcal{Q}_{V,D^\bullet}$ defined in Section 7.1. Let ${}_{i,n}\mathcal{P}_{V,D^\bullet}$ be the subset of $\mathcal{P}_{V,D^\bullet}$ consisting of all objects K such that

$$\mathrm{Supp}(\Phi_\Omega^{{}_i\Omega}(K)) \subset {}_{i,\geq n}\mathbf{E}_{{}_i\Omega}(V,D) \quad \text{and} \quad \mathrm{Supp}(\Phi_\Omega^{{}_i\Omega}(K)) \cap {}_{i,n}\mathbf{E}_{{}_i\Omega}(V,D) \neq \emptyset$$

where $\Phi_\Omega^{{}_i\Omega}$ is the Fourier transform. We set

$$\varepsilon_i(K) = n, \quad \text{if } K \in {}_{i,n}\mathcal{P}_{V,D^\bullet}. \tag{62}$$

Lemma 7.2.1. *For any $K \in {}_{i,n}\mathcal{P}_{V,D^\bullet}$, there exists a unique $\bar{K} \in {}_{i,0}\mathcal{P}_{W,D^\bullet}$, with $\dim W = \dim V - ni$, such that the following statements hold.*

(1) *${}_i\mathcal{R}^{(n)}(K) = \bar{K} \oplus \bar{K}'$, where $\bar{K}'$ consists of simple perverse sheaves, with possible shifts, in ${}_{i,\geq 1}\mathcal{P}_{W,D^\bullet}$ and ${}_i\mathcal{R}^{(n)}$ is defined in Section 5.4, (46).*

(2) *$\mathcal{F}_i^{(n)}(\bar{K}) = K \oplus K'$, where K' consists of simple perverse sheaves, with possible shifts, in ${}_{i,\geq n+1}\mathcal{P}_{V,D^\bullet}$ and $\mathcal{F}_i^{(n)}$ is defined in Section 5.4, (46).*

(3) *The assignment $K \mapsto \bar{K}$ defines a bijection ${}_i\tilde{\mathcal{R}}^{(n)} : {}_{i,n}\mathcal{P}_{V,D^\bullet} \to {}_{i,0}\mathcal{P}_{W,D^\bullet}$.*

(4) *The assignment* $\bar{K} \mapsto K$ *defines a bijection* $\tilde{\mathcal{F}}_i^{(n)} : {}_{i,0}\mathcal{P}_{W,D^\bullet} \to {}_{i,n}\mathcal{P}_{V,D^\bullet}$*, inverse to* ${}_i\tilde{\mathcal{R}}^{(n)}$.

The proof is similar to the proofs of Lemma 3.2.10 in [Zh08] and Lemma 6.4 in [L91].

Proposition 7.2.2. *For any* $K \in \mathcal{Q}_{V,D^\bullet}$*, there exist complexes* $M_1, \cdots, M_k$ *and* $N_1, \cdots, N_l$ *of the form* $L_{\underline{\mathbf{a}}}$ *for some* $\underline{\mathbf{a}}$*, up to some shift, such that*

$$K \oplus M_1 \oplus \cdots \oplus M_k = N_1 \oplus \cdots \oplus N_l.$$

The proof is similar to the proofs of Proposition 3.2.6 in [Zh08] and Proposition 7.3 in [L91].

7.3. **Singular supports of the objects in** $\mathcal{P}_{V,D^\bullet}$**.** Let $\tilde{X}_{\underline{\mathbf{a}}}$ be the variety of all pairs $(X, F^\bullet)$, where $X \in \mathbf{\Lambda}_{V,D}$ and $F^\bullet$ is a flag of type $\underline{\mathbf{a}}$, such that $F^\bullet$ is X-invariant. Then we have a natural projection

$$\tilde{X}_{\underline{\mathbf{a}}} \to \mathbf{\Lambda}_{V,D}.$$

We denote by $\tilde{Y}_{\underline{\mathbf{a}}}$ the image of $\tilde{X}_{\underline{\mathbf{a}}}$ under this projection. Similar to the proof of Theorem 13.3 and Corollary 13.6 in [L91], one can show the following proposition.

Proposition 7.3.1. $\mathrm{SS}(L_{\underline{\mathbf{a}}}) \subseteq \tilde{Y}_{\underline{\mathbf{a}}} \subseteq \mathbf{\Lambda}_{V,D}$.

Given any pair $(X, F^\bullet)$ in $\tilde{X}_{\underline{\mathbf{a}}}$, we have that both F^{m^1} and F^{m^1+1} are $X = (x, p, q)$-invariant. Write $F^{m^1} = U \oplus D^0$ and $F^{m^1+1} = U \oplus D^1$, we have $p(D^0) \subseteq U$ and $q(U) \subseteq D^1$. This implies that $p(D^0) \subseteq U \subseteq q^{-1}(D^1)$. Since U is x-invariant, we have $\overline{p(D^0)} \subseteq U \subseteq \underline{q^{-1}(D^1)}$. Since $F^{m^1+m^2+1}$ is X-invariant, we have immediately $\overline{p(D^1)} = 0$. So given any pair $(X, F^\bullet) \in \tilde{X}_{\underline{\mathbf{a}}}$, the element X satisfies the condition (26). In other words, $\tilde{Y}_{\underline{\mathbf{a}}} \subseteq \mathbf{\Pi}_{V,D^\bullet}$. By combining with Proposition 7.3.1, we have

Proposition 7.3.2. $\mathrm{SS}(L_{\underline{\mathbf{a}}}) \subseteq \tilde{Y}_{\underline{\mathbf{a}}} \subseteq \mathbf{\Pi}_{V,D^\bullet} \subseteq \mathbf{\Lambda}_{V,D}$.

7.4. **The map** $Y_\bullet$**.** First, we assume that $D^2 = 0$ in the set up of Section 7.1. In this case, the complex $L_{\underline{\mathbf{a}}}$ defined in (59) is a special case of the complex $L_{\mathbf{i},\mathbf{a}}$ defined in (51) for the graph $\tilde{\Gamma}$. Also, $\mathbf{\Pi}_{V,D^\bullet} \subseteq \mathbf{\Lambda}_{V\oplus D}$. Thus, we can use Theorem 6.2.2 (2) in [KS97], [K07] and Remark 4.27 in [Sch09] to define a map

$$Y_\bullet : \mathcal{P}_{V,D^\bullet} \to \mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}, \quad K \mapsto Y_K, \tag{63}$$

such that

$$\begin{aligned} &Y_K \subseteq \mathrm{SS}(K) \subseteq Y_K \cup \cup_{K' : \varepsilon_i(K') \geq \varepsilon_i(K)} Y_{K'}, \\ &\varepsilon_i(K) = \varepsilon_i(Y_K), \quad \forall i \in I, \\ &K \neq K' \quad \text{implies} \quad Y_K \neq Y_{K'}. \end{aligned}$$

The last condition means that $Y_\bullet$ is an injective map. Moreover, $Y_\bullet$ is surjective hence bijective.

To show that $Y_\bullet$ is bijective, it is reduced to show that the two sets $\mathcal{P}_{V,D^\bullet}$ and $\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}$ are of the same size. This can be argued as follows. Let $\mathbf{V}_1$ be the space over $\mathbb{C}$ spanned by the elements in $\mathcal{P}_{V,D^\bullet}$. Let $\mathbf{V}_2$ be the space over $\mathbb{C}$ spanned by the complexes $L_{\underline{\mathbf{a}}}$ for various $\underline{\mathbf{a}}$. Then, modulo specializing the shift functor to 1, we have by Proposition 7.2.2 that

$$\mathbf{V}_1 \simeq \mathbf{V}_2$$

as vector spaces over $\mathbb{C}$.

On the other hand, let $\mathbf{W}_2$ be the space spanned by the constructible functions $L'_{\underline{\mathbf{a}}}$ on $\mathbf{\Pi}_{V,D^\bullet}$ defined in a similar way as $L_{\underline{\mathbf{a}}}$. Let $\mathbf{W}_1$ be the vector space over $\mathbb{C}$ spanned by the semicanonical basis elements f_Y defined in [L00a] such that $Y \subseteq \mathbf{\Pi}_{V,D^\bullet}$. Then we have (We refer the interested reader to the paper [L00a] and the reference therein for the precise definitions of $L'_{\underline{\mathbf{a}}}$ and f_Y.)

$$\mathbf{W}_1 = \mathbf{W}_2.$$

Indeed, it is clear that $\mathbf{W}_2 \subseteq \mathbf{W}_1$. Suppose that $f_Y \in \mathbf{W}_1$ satisfies that $\varepsilon_i(Y) = 0$ for any $i \in I$. It is clear that $f_Y = L'_d \cdot f_{\bar{Y}}$ for some $\bar{Y} \in \mathbf{\Lambda}_V$ where L'_d is the linear map defined in analog with the functor L_d. From this, we see that $f_Y \in \mathbf{W}_2$. Suppose that $f_Y \in \mathbf{W}_1$ such that $r = \varepsilon_i(Y) > 0$ for some $i \in I$. We prove by induction that $f_Y \in \mathbf{W}_2$. We assume that $f_{Y'} \in \mathbf{W}_2$ if $Y \subseteq \mathbf{\Pi}_{V',D^\bullet}$ for any proper subspace $V' \subset V$ and $f_Z \in \mathbf{W}_2$ if $\varepsilon_i(Z) > \varepsilon_i(Y)$.

By [L00a, 2.9 (b)], we have an irreducible component Y' such that

$$f_i^r \cdot f_{Y'} = f_Y + \sum_{Z:\varepsilon_i(Z)>\varepsilon_i(Y)} c_{Y,Z} f_Z, \quad \text{for some} \quad c_{Y,Z} \in \mathbb{Z},$$

where f_i^r is defined in analog with the functor $\mathcal{F}_i^{(r)}$. Since $Y \subseteq \mathbf{\Pi}_{V,D^\bullet}$, we have $Y' \subseteq \mathbf{\Pi}_{V',D^\bullet}$ for some V'. Hence, $Z \subseteq \mathbf{\Pi}_{V,D^\bullet}$. By the above identity and induction assumption, we see that $f_Y \in \mathbf{W}_2$. So $\mathbf{W}_1 \subseteq \mathbf{W}_2$. We have finished the proof of $\mathbf{W}_1 = \mathbf{W}_2$.

By [L91] and [L00a], we see that

$$\mathbf{V}_2 = \mathbf{W}_2$$

because they correspond to the same subspace in U^-, the space obtained from $\mathbf{U}^-$ by specializing v at 1. By summing up the above analysis, we have

$$\#\mathcal{P}_{V,D^\bullet} = \dim \mathbf{V}_1 = \dim \mathbf{V}_2 = \dim \mathbf{W}_2 = \dim \mathbf{W}_1 = \#\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}.$$

This shows that the maps $Y_\bullet$ is surjective. Altogether, we have

Lemma 7.4.1. *When $D^2 = 0$, the map $Y_\bullet$ is bijective.*

Second, we assume that $D^2 \neq 0$. We write $\mathcal{P}_{V,D^1}$ (resp. $\mathbf{\Pi}_{V,D^1}$) for $\mathcal{P}_{V,D^\bullet}$ (resp. $\mathbf{\Pi}_{V,D^\bullet}$) for the case when $D^2 = 0$. The fully faithful functor $\cdot L_{d^2}$ defines a bijection

$$L_{d^2} : \mathcal{P}_{V,D^1} \to \mathcal{P}_{V,D^\bullet},$$

such that $\varepsilon_i(\tilde{K}) = \varepsilon_i(\tilde{K} \cdot L_{d^2})$ for any $i \in I$. Similarly, we can define a bijection between the sets $\mathrm{Irr}\, \mathbf{\Pi}_{V,D^1}$ and $\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}$. It is clear from the construction that the two bijections are compatible. Therefore, we can define a similar map $Y_\bullet : \mathcal{P}_{V,D^\bullet} \to \mathbf{\Pi}_{V,D^\bullet}$ satisfying the same property as that of (63).

Summing up the above analysis, we have the following proposition.

Proposition 7.4.2. *We have a bijective map $Y_\bullet : \mathcal{P}_{V,D^\bullet} \to \mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}$, $K \mapsto Y_K$ such that*

$$Y_K \subseteq \mathrm{SS}(K) \subseteq Y_K \cup \cup_{K':\varepsilon_i(K')\geq\varepsilon_i(K)} Y_{K'} \quad \textit{and} \quad \varepsilon_i(K) = \varepsilon_i(Y_K), \quad \forall i \in I.$$

Remark 7.4.3. We shall present a second proof of Proposition 7.4.2 in Section 7.5, which can be generalized to the general case in Section 8.

For any $K \in \mathcal{P}_{V,D^\bullet}$ and $i \in I$, we define

$$\mathrm{wt}(K) = \lambda - \nu \in \mathbf{X}, \quad \varepsilon_i(K) = n, \quad \text{if } K \in {}_{i,n}\mathcal{P}_{V,D^\bullet}, \quad \varphi_i = \varepsilon_i(K) + (i, \mathrm{wt}(K)),$$

where $\lambda \in \mathbf{X}$ is a fixed element such that $\lambda(i) = d_i$, for any $i \in I$, and $\nu \in \mathbf{X}$ is via the imbedding $\mathbb{N}[I] \to \mathbf{X}$. We also define, for any $K \in \mathcal{P}_{V,D^\bullet}$,

$$\tilde{f}_i(K) = \tilde{\mathcal{F}}_i^{(\varepsilon_i(K)+1)}\, {}_i\tilde{\mathcal{R}}^{(\varepsilon_i(K))}(K) \quad \text{and} \quad \tilde{e}_i(K) = \begin{cases} \tilde{\mathcal{F}}_i^{(\varepsilon_i(K)-1)}\, {}_i\tilde{\mathcal{R}}^{(\varepsilon_i(K))}(K) & \text{if } \varepsilon_i(K) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The above maps $(\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)_{i\in I}$ define a crystal structure on

$$\mathcal{P}_{D^\bullet} = \sqcup_{\nu\in\mathbb{N}[I]} \mathcal{P}_{V(\nu),D^\bullet}.$$

Moreover, it is clear that the crystal $\mathcal{P}_{D^\bullet}$ is generated by the objects K such that $\varepsilon_i(K) = 0$ for any $i \in I$.

The assignment $K \mapsto Y_K$ in (7.4.2) defines a strict crystal isomorphism

$$Y_\bullet : \mathcal{P}_{D^\bullet} \to \mathrm{Irr}\,(D^\bullet).$$

From this isomorphism and Theorem 4.6.2, we have

Theorem 7.4.4. *The crystal structure on $\mathcal{P}_{D^\bullet}$ is isomorphic to the crystal structure of $B(\lambda^1) \otimes B(\lambda^2, \infty)$ via $\psi Y_\bullet$ where ψ is defined in (34) .*

7.5. **A filtration on $\mathcal{P}_{V,D^\bullet}$.** Let

$$\Xi_{V,D^\bullet} = \{K \in \mathcal{P}_{V,D^\bullet} | \ \varepsilon_i(K) = 0, \quad \forall i \in I\} \quad \text{and} \quad \Xi_{D^\bullet} = \sqcup_{\nu\in\mathbb{N}[I]} \Xi_{V(\nu),D^\bullet}.$$

Lemma 7.5.1. *We have $K \in \Xi_{V,D^\bullet}$ if and only if K is of the form $L_{d^1} K_b L_{d^2}$ where K_b is the simple perverse sheaf corresponding to an element b in $B(\lambda^1)_\nu$ with $\nu = \dim V$.*

Proof. Suppose that $K \in \Xi_{V,D^\bullet}$. Then it is clear that K is a direct summand of a certain complex $L_{d^1(\mathbf{i},a)d^2} = L_{d^1} \cdot L_{(\mathbf{i},\mathbf{a})} \cdot L_{d^2}$, up to a shift. Since $L_{d^1}\cdot$ and $\cdot L_{d^2}$ are fully faithful functors, we see that K has to be of the form $L_{d^1} K_b L_{d^2}$ for some simple perverse sheaf in $\mathcal{P}_V$. If K_b is a direct summand of the complex $L_{(\mathbf{i},\mathbf{a})(i,d^1_i+1)}$, then it is clear that $\varepsilon_i(K) > 0$. This shows that K_b is a simple perverse sheaf such that $b \in B(\lambda^1)_\nu$. It is clear that if $K = L_{d^1} K_b L_{d^2}$ such that $b \in B(\lambda^1)_\nu$ then $K \in \Xi_{V,D^\bullet}$. $\square$

Note that in general, the set $\Xi_{D^\bullet}$ has infinitely many elements. Let us order the elements in $\Xi_{D^\bullet}$ in a way, say

$$\xi_1, \xi_2, \cdots, \xi_n, \cdots,$$

such that if $\xi_m \in \Xi_{V,D^\bullet}$ and $\xi_n \in \Xi_{W,D^\bullet}$ with W a proper subspace of V, then $n < m$.

Let $\mathcal{Q}^{\leq n}_{V,D^\bullet}$ be the full subcategory of $\mathcal{Q}_{V,D^\bullet}$ whose simple objects are direct summands of the complexes $L_{(\mathbf{i},\mathbf{a})} \cdot \xi_m[z]$ in $\mathcal{Q}_{V,D^\bullet}$, for $z \in \mathbb{Z}$ and $1 \leq m \leq n$. Then we have a filtration of $\mathcal{Q}_{V,D^\bullet}$:

$$\mathcal{Q}^{\leq 1}_{V,D^\bullet} \subset \mathcal{Q}^{\leq 2}_{V,D^\bullet} \subset \cdots \subset \mathcal{Q}^{\leq n}_{V,D^\bullet} \subset \cdots. \tag{64}$$

Note that $\mathcal{Q}^{\leq n}_{V,D^\bullet} = \mathcal{Q}^{\leq m}_{V,D^\bullet} = \mathcal{Q}_{V,D^\bullet}$ for large enough m and n.

Similarly, we have a filtration for the simple objects in $\mathcal{Q}_{V,D^\bullet}$:

$$\mathcal{P}^{\leq 1}_{V,D^\bullet} \subset \mathcal{P}^{\leq 2}_{V,D^\bullet} \subset \cdots \subset \mathcal{P}^{\leq n}_{V,D^\bullet} \subset \cdots,$$

where $\mathcal{P}^{\leq n}_{V,D^\bullet}$ are subsets of $\mathcal{P}_{V,D^\bullet}$ whose element appears in $\mathcal{Q}^{\leq n}_{V,D^\bullet}$. We set

$$\mathcal{P}^{n}_{V,D^\bullet} = \mathcal{P}^{\leq n}_{V,D^\bullet} \backslash \mathcal{P}^{\leq n-1}_{V,D^\bullet}, \quad \forall n \in \mathbb{N}.$$

(The filtration comes from the one in [BGG71], [BGG75], [BGG76] and [BG80].)

Lemma 7.5.2. *We have $\xi_n \in \mathcal{P}^n_{V,D^\bullet}$.*

Proof. Assume that $\xi_n \in \mathcal{Q}^{\leq n-1}_{V,D^\bullet}$. Since $\varepsilon_i(\xi_n) = 0$ for any $i \in I$, ξ_n can only be expressed as a direct sum whose simple summands are from the set $\{\xi_1, \cdots, \xi_{n-1}\}$. This contradicts with the fact that the set $\{\xi_1, \cdots, \xi_{n-1}, \xi_n\}$ is linearly independent, because it is a subset of the canonical basis $B(\lambda^1)$ by Lemma 7.5.1. Therefore, $\xi_n \in \mathcal{P}^n_{V,D^\bullet}$. □

Recall from [KS97, Theorem 6.2.2] that there exists a crystal isomorphism

$$Y_\bullet : \sqcup_V \mathcal{P}_V \to \mathrm{Irr}\, \mathbf{\Lambda}_V,$$

where V runs over a set of representatives of the isomorphism classes of the I-graded vector spaces, such that

$$Y_K \subseteq \mathrm{SS}(K) \subseteq Y_K \cup \cup_{\varepsilon_i(Y_K) \geq \varepsilon_i(Y_{K'})} Y_{K'}, \quad \forall i \in I.$$

Let Y_{ξ_n} be the irreducible component in $\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}$ obtained from Y_{K_b} if $\xi_n = L_{d^1} K_b L_{d^2}$. Let $\mathbf{\Pi}^n_{V,D^\bullet}$ be the subset in $\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}$ defined in a similar manner as $\mathcal{P}^n_{V,D^\bullet}$ to $\mathcal{P}_{V,D^\bullet}$. In other words, $\mathbf{\Pi}_{V,D^\bullet}$ is the crystal generated by the element Y_{ξ_n}.

Let π'_a be the similar morphisms to π_a in (42) for $a = 1, 2, 3$ with the varieties '$\mathbf{E}$' replaced by the varieties '$\mathbf{\Pi}$'. For any irreducible component $\bar{Y} \in \mathrm{Irr}\, \mathbf{\Lambda}_T$, we set

$$\bar{Y} \cdot Y_{\xi_n} = \pi'_3 \pi'_2 (\pi'_1)^{-1} (\bar{Y} \times Y_{\xi_n})$$

to be the closed subvariety of $\mathbf{\Pi}_{V,D^\bullet}$, where we assume that $Y_{\xi_n} \in \mathrm{Irr}\, \mathbf{\Pi}_{W,D^\bullet}$ and $V = T \oplus W$. Let $Y^0_{\xi_n}$ be the open subvariety of Y_{ξ_n} consisting of all points X such that $\varepsilon_j(X) = 0$ for any $j \in I$. We claim that

- the closure, Y, of $\bar{Y} \cdot Y^0_{\xi_n}$ is an irreducible component in $\mathbf{\Pi}_{V,D^\bullet}$ such that $\varepsilon_j(Y) = \varepsilon_j(\bar{Y})$ for any $j \in I$.

Indeed, the restriction of π'_1 to $(\pi'_1)^{-1}(\bar{Y} \times Y^0_{\xi_n})$ is smooth with connected fiber. This can be proved in a similar manner as the proof of Lemma 4.4.1 due to the fact that the condition that $X(i)$ is injective is due to the condition that $(i)X$ is surjective. Moreover, the restriction of π'_3 to $\pi'_2(\pi_1)'^{-1}(\bar{Y} \cdot Y^0_{\xi_n})$ is injective. This is due to the fact that the space U such that $D \oplus U$ is X-stable for any $X = (x, p, q) \in \pi'_2(\pi_1)'^{-1}(\bar{Y} \cdot Y^0_{\xi_n})$ is $\overline{p(D)}$ by the definition of $Y^0_{\xi_n}$. The above analysis implies that $\bar{Y} \cdot Y_{\xi_n}$ is irreducible, of the desired dimension and $\varepsilon_j(\bar{Y} \times Y_{\xi_n}) = \varepsilon_j(\bar{Y})$ for any $j \in I$. The claim follows.

It is clear that

$$Y \subseteq \bar{Y} \cdot Y_{\xi_n} \subset Y \cup \cup_{\varepsilon_i(Y') \geq \varepsilon_i(Y)} Y', \quad \forall i \in I.$$

Proposition 7.5.3. *The assignment $\bar{Y} \mapsto Y$, where Y is the closure of $\bar{Y} \cdot Y^0_{\xi_n}$, defines a bijection*

$$\gamma : \mathrm{Irr}\, \mathbf{\Lambda}_T \to \mathrm{Irr}\, \mathbf{\Pi}^n_{V,D^\bullet},$$

which is compatible with the actions $\tilde{e}^r_i$ and $\tilde{f}^r_i$ for any $i \in I$ and $r \in \mathbb{N}$.

Proof. It is clear that γ is injective. γ is surjective follows from Theorem 4.6.2. It can also be proved by induction with respect to the dimension of T. The compatibility of γ with the actions $\tilde{e}_i^r$ and $\tilde{f}_i^r$ is reduced to the compatibility of the diagram (31) and the diagram similar to (42). □

Proposition 7.5.4. *Suppose that $\xi_n \in \Xi_{W,D^\bullet}$. Given any $\bar{K} \in \mathcal{P}_T$, there is a unique $K \in \mathcal{P}^n_{V,D^\bullet}$, with $V \simeq T \oplus W$, such that*

$$\begin{aligned}
&\bar{K} \cdot \xi_n = K \oplus M, \quad \textit{where } M \in \mathcal{Q}^{\leq n}_{V,D^\bullet};\\
(65)\qquad &\varepsilon_j(\bar{K}) = \varepsilon_j(K), \quad \varepsilon_j(K) \leq \varepsilon_j(M), \quad \forall j \in I; \quad \varepsilon_i(K) < \varepsilon_i(M), \quad \textit{for some } i \in I;\\
&\gamma(Y_{\bar{K}}) \subseteq \mathrm{SS}(K) \subseteq \gamma(Y_{\bar{K}}) \cup \cup_{\varepsilon_j(Y') \geq \varepsilon_j(K)} Y', \quad \forall j \in I.
\end{aligned}$$

Moreover, all elements in $\mathcal{P}^n_{V,D^\bullet}$ are obtained in this way. The assignment $\bar{K} \mapsto K$ in (65) defines a bijection $\mathcal{P}_T \to \mathcal{P}^n_{T\oplus W,D^\bullet}$.

Proof. We shall prove the statement by induction with respect to the dimension of T. The statement is clear when $T = 0$ by Lemma 7.5.2 and the choice of Y_{ξ_n}. Suppose that the statement holds for any proper subspaces in a non zero vector space T. Then there exists a vertex $i \in I$ such that $\varepsilon_i(\bar{K}) > 0$. Set $r = \varepsilon_i(\bar{K}) > 0$. Then, by Lemma 7.2.1, there exists a simple perverse sheaf $\bar{L} \in \mathcal{P}_{\bar{T}}$ for some subspace $\bar{T}$ in T with $\varepsilon_i(\bar{L}) = 0$ such that

$$\mathcal{F}_i^{(r)} \cdot \bar{L} = \bar{K} \oplus \bar{M}, \quad \text{where } \varepsilon_i(\bar{M}) > r.$$

By induction, we have

$$\bar{L} \cdot \xi_n = L \oplus N, \quad \text{where } L \in \mathcal{P}^n_{\bar{V},D^\bullet} \text{ and } N \in \mathcal{Q}^{\leq n}_{\bar{V},D^\bullet},$$

for $\bar{V}$ a certain proper subspace in V. Moreover, the complexes L and N satisfy $\varepsilon_i(L) = 0$ and $\varepsilon_i(N) > 0$ and $\varepsilon_j(N) \geq \varepsilon_j(L)$ for any $j \in I$. From this and by Lemma 7.2.1, we see that there is a unique simple perverse sheaf K in $\mathcal{F}_i^{(r)} \cdot \bar{L} \cdot \xi_n$ such that $\varepsilon_i(K) = r$ and that K is a direct summand of $\mathcal{F}_i^{(r)} \cdot L$. Observe that

$$\mathcal{F}_i^{(r)} L \oplus \mathcal{F}_i^{(r)} N = \mathcal{F}_i^{(r)} \cdot \bar{L} \cdot \xi_n = \bar{K} \cdot \xi_n \oplus \bar{M} \cdot \xi_n,$$

and $\varepsilon_i(\bar{M} \cdot \xi_n), \varepsilon_i(\mathcal{F}_i^{(r)} N) > r$. We see that K has to be a direct summand in $\bar{K} \cdot \xi_n$, i.e.,

$$\bar{K} \cdot \xi_n = K \oplus M, \quad \text{for some } M \in \mathcal{Q}^{\leq n}_{V,D^\bullet} \text{ such that } \varepsilon_i(M) > r.$$

We are left to show that $\varepsilon_j(\bar{K}) = \varepsilon_j(K)$ for any $j \in I$ and the claim on their singular supports. (This will automatically imply that $\varepsilon_j(K) \leq \varepsilon_j(M)$ for any $j \in I$.) By the induction assumption, we have

$$\gamma(Y_{\bar{L}}) \subseteq \mathrm{SS}(L).$$

This implies that

$$\gamma(Y_{\bar{K}}) = \tilde{f}_i^r \gamma(Y_{\bar{L}}) \subseteq \mathrm{SS}(\mathcal{F}_i^{(r)} \cdot L),$$

where the equality is due to Proposition 7.5.3. Now the complex K is the only one in $\mathcal{F}_i^{(r)} \cdot L$ such that $\varepsilon_i(K) = r$ and the evaluations of ε_i at all other simple summands have values strictly larger than r. So

$$\gamma(Y_{\bar{K}}) \subseteq \mathrm{SS}(K).$$

This implies that $\varepsilon_j(K) = \varepsilon_j(Y(\bar{K})) = \varepsilon_j(\bar{K})$ for any $j \in I$.

Since $L \in \mathcal{P}^n_{V,D^\bullet}$, so is K. Otherwise, it will lead to a contradiction by Lemma 7.2.1. The statement (65) follows.

Now we show that all elements K in $\mathcal{P}^n_{V,D^\bullet}$ can be obtained in the above way. If $\varepsilon_i(K) = 0$ for all $i \in I$, then $K = \xi_n$. The proof is trivial. If there is an i such that $\varepsilon_i(K) \neq 0$, then we can use a similar argument as above to show that there is a complex $\bar{K} \in \mathcal{P}_T$ for some T such that condition (65) holds. $\square$

By combining Propositions 7.5.3 and 7.5.4, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{P}_T \arrow[r] \arrow[d] & \mathcal{P}^n_{V,D^\bullet} \arrow[d, "Y_\bullet"] \\
\mathbf{\Pi}_V \arrow[r] & \mathbf{\Pi}^n_{V,D^\bullet},
\end{tikzcd}$$

where $Y_\bullet(K) = \gamma(Y_{\bar{K}})$. Since all three unnamed maps are bijective, we see that $Y_\bullet$ is bijective. This is a second proof of Proposition 7.4.2.

7.6. **Grothendieck group $\mathcal{K}_{D^\bullet}$.** Let ${}_{\mathbb{A}}\mathcal{K}_{D^\bullet}$ be the $\mathbb{A}$-module with basis $\mathcal{P}_{D^\bullet}$. Let $\mathcal{K}_{D^\bullet}$ be the $\mathbb{Q}(v)$-vector space $\mathbb{Q}(v) \otimes_{\mathbb{A}} {}_{\mathbb{A}}\mathcal{K}_{D^\bullet}$. The functors $\mathcal{F}_i^{(n)}$ and ${}_i\mathcal{R}^{(n)}$ and $\mathcal{R}_i^{(n)}$ defined in Section 5.4 descend to linear maps on $\mathcal{K}_{D^\bullet}$. We shall use the same letters to denote these linear maps.

From these linear maps, we can define a $\mathbf{U}$-module structure on $\mathcal{K}_{D^\bullet}$, and a ${}_{\mathbb{A}}\mathbf{U}$-module structure on ${}_{\mathbb{A}}\mathcal{K}_{D^\bullet}$. The functors $\mathcal{F}_i^{(n)}$ correspond to the action $F_i^{(n)}$. By (10) and the fact that ${}_i\bar{r}$ and $\bar{r}_i$ gets identified with ${}_i\mathcal{R}$ and $\mathcal{R}_i$ in [L93], the E_i-action on $\mathcal{K}_{D^\bullet}$ is defined by

$$E_i(x) = \frac{\mathcal{R}_i(x) - v^{-(i,-|x|+i)}\, {}_i\mathcal{R}(x)}{v - v^{-1}}, \quad \forall x \in \mathcal{K}_{D^\bullet}. \tag{66}$$

It can also be expressed as

$$E_i(x) = (\mathbf{D}v^{(i,|x|-i)}\, {}_i\mathcal{R}(\mathbf{D}(x)) - v^{(i,|x|-i)}\, {}_i\mathcal{R}(x))/(v - v^{-1}), \quad \forall x \in \mathcal{K}_{D^\bullet}, \tag{67}$$

where $\mathbf{D}$ is the Verdier duality functor. The action K_i is corresponding to the shift functors $P \mapsto P[(i, \lambda - \nu)]$ in $\mathcal{Q}_{V,D^\bullet}$. By (10), we see that the module $\mathcal{K}_{D^\bullet}$ equipped with the defined actions is a $\mathbf{U}$-module. Moreover, we have

Theorem 7.6.1. *There are isomorphisms*

$$ {}_{\mathbb{A}}M_{\lambda^2} \otimes {}_{\mathbb{A}}V_{\lambda^1} \simeq {}_{\mathbb{A}}\mathcal{K}_{D^\bullet}, \quad M_{\lambda^2} \otimes V_{\lambda^1} \simeq \mathcal{K}_{D^\bullet}, \tag{68}$$

as ${}_{\mathbb{A}}\mathbf{U}$-modules and $\mathbf{U}$-modules, respectively.

Proof. By an argument similar to (18), we have a surjective homomorphism of $\mathbf{U}$-modules:

$$M_{\lambda^2} \otimes V_{\lambda^1} \twoheadrightarrow \mathcal{K}_{D^\bullet}.$$

By Theorem 7.4.4, the spaces $M_{\lambda^2} \otimes V_{\lambda^1}$ and $\mathcal{K}_{D^\bullet}$ have the same dimension at each level. So the above morphism is an isomorphism. The result over $\mathbb{A}$ is proved in a similar way. Theorem follows. $\square$

Under the isomorphism in the above Theorem, we see that $\mathcal{P}_{D^\bullet}$ is a basis of $M_{\lambda^2} \otimes V_{\lambda^1}$. We shall show in the following that it is the same as the *canonical basis* of $M_{\lambda^2} \otimes V_{\lambda^1}$ defined in an algebraic way.

The involution $\mathbf{D} : \mathcal{K}_{D^\bullet} \to \mathcal{K}_{D^\bullet}$ is compatible with the $\mathbf{U}$ action on $\mathcal{K}_{D^\bullet}$ in the sense that we have

$$\mathbf{D}(E_i x) = E_i(\mathbf{D}(x)), \quad \mathbf{D}(F_i x) = F_i(\mathbf{D}(x)), \quad \mathbf{D}(K_i x) = K_{-i}\mathbf{D}(x),$$

for any $i \in I$, $x \in \mathcal{K}_{D^\bullet}$. Indeed, the compatibility of $\mathbf{D}$ with E_i follows from (67) and the compatibility of $\mathbf{D}$ with F_i and K_i is obvious.

Let

$$\Psi : M_{\lambda^2} \otimes V_{\lambda^1} \to M_{\lambda^2} \otimes V_{\lambda^1},$$

be the unique involution defined in [L93, 27.3.1] such that

$$\Psi(fx) = \bar{f}\bar{x}, \quad \Psi(E_i x) = E_i \bar{x}, \quad \Psi(F_i x) = F_i \bar{x} \quad \text{and} \quad \Psi(K_i x) = K_{-i}\bar{x},$$

for any $f \in \mathbb{Q}(v)$, $x \in M_{\lambda^2} \otimes V_{\lambda^1}$ and $i \in I$, where $x \mapsto \bar{x}$ is the bar involution defined as the tensor product of bar involutions on M_{λ^2} and V_{λ^1}, respectively.

Corollary 7.6.2. *We have the following commutative diagram:*

$$\begin{array}{ccc} M_{\lambda^2} \otimes V_{\lambda^1} & \longrightarrow & \mathcal{K}_{D^\bullet} \\ {\scriptstyle\Psi}\big\downarrow & & {\scriptstyle\mathbf{D}}\big\downarrow \\ M_{\lambda^2} \otimes V_{\lambda^1} & \longrightarrow & \mathcal{K}_{D^\bullet}, \end{array}$$

where the horizontal morphisms are from (68).

This can be proved by two steps. The first one is observed that the above diagram commutes for any elements of the form $x\xi_{\lambda^1} \otimes \xi_{\lambda^2}$ in $M_{\lambda^2} \otimes V_{\lambda^1}$. The second one is that the diagram commutes for the rest elements can be proved by induction because the involutions Ψ and $\mathbf{D}$ are compatible with the $\mathbf{U}$-actions.

Recall from [L93, 1.2.3], we have a unique symmetric bilinear form on $\mathbf{U}^-$ subject to the following properties:

$$(1,1) = 1, \quad (F_i, F_j) = \delta_{ij}\frac{1}{1 - v^{-2}}, \quad (F_i x, y) = \frac{1}{1 - v^{-2}}(x,\ {}_i r(y)),$$

for any $i, j \in I$ and $x, y \in \mathbf{U}^-$, where ${}_i r : \mathbf{U}^- \to \mathbf{U}^-$ is a linear map defined by

$${}_i r(1) = 0,\ {}_i r(F_j) = \delta_{ij},\ {}_i r(xy) = {}_i r(x)y + v^{|x|\cdot i} x\, {}_i r(y), \quad x, y \text{ homogeneous}.$$

As vector spaces, $\mathbf{U}^-$ is isomorphic to M_λ. So the bilinear form transports to M_λ. Recall from [L93, 19.1.1], we have a unique symmetric bilinear form on V_λ subject to the following properties:

$$(\xi_\lambda, \xi_\lambda) = 1, \quad (E_i x, y) = (x, vK_i F_i y), \quad (K_i x, y) = (x, K_i y), \quad (F_i x, y) = (x, vK_i^{-1} E_i y),$$

for any $x, y \in V_\lambda$ and $i \in I$. We define a bilinear form on $M_{\lambda^2} \otimes V_{\lambda^1}$ by $(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2, y_2)$ for any $x_1, y_1 \in M_{\lambda^2}$ and $x_2, y_2 \in V_{\lambda^1}$. It is straightforward to show that the bilinear form on $M_{\lambda^2} \otimes V_{\lambda^1}$ satisfies the following properties:

$$\begin{aligned} &(\xi_{\lambda^2} \otimes \xi_{\lambda^1}, \xi_{\lambda^2} \otimes \xi_{\lambda^1}) = 1, \\ &(\phi_i(x_1 \otimes x_2), y_1 \otimes y_2) = \frac{1}{1 - v^{-2}}(x_1 \otimes x_2, \epsilon_i(y_1 \otimes y_2)), \end{aligned}$$

for any $x_1, y_1 \in M_{\lambda^2}$, $x_2, y_2 \in V_{\lambda^1}$, where ϕ_i and ϵ_i, for any $i \in I$, are defined by

$$\phi_i(x_1 \otimes x_2) = F_i x_1 \otimes K_i^{-1} x_2 + x_1 \otimes F_i x_2,$$
$$\epsilon_i(x_1 \otimes x_2) = {}_i r(x_1) \otimes K_i^{-1} x_2 + (v - v^{-1}) x_1 \otimes K_i^{-1} E_i x_2, \quad \forall x_1 \otimes x_2 \in M_{\lambda^2} \otimes V_{\lambda^1}.$$

Similar to [L93, 13.1.6-13.1.12], we can define geometrically a bilinear form on $\mathcal{K}_{D^\bullet}$:

$$(,) : \mathcal{K}_{D^\bullet} \times \mathcal{K}_{D^\bullet} \to \mathbb{Q}(v),$$

subject to the following properties:

$$(L_d, L_d) = 1, \quad (K, K') \in \delta_{K,K'} + v^{-1}\mathbb{Z}[[v^{-1}]] \cap \mathbb{Q}(v), \quad \forall K, K' \in \mathcal{P}_{D^\bullet},$$
$$(F_i K, K') = \frac{1}{1 - v^{-2}} (K, {}_i\bar{\mathcal{R}}(K')), \quad \forall K, K' \in \mathcal{P}_{D^\bullet},$$

where ${}_i\bar{\mathcal{R}} = \mathbf{D} \circ {}_i\mathcal{R} \circ \mathbf{D}$, which gets identified with the linear map ${}_i r$ on $\mathbf{U}^-_{\tilde{\Gamma}}$.

It is clear that the data $(\phi_i, \epsilon_i | i \in I)$ are compatible with the data $(F_i, {}_i\bar{\mathcal{R}} | i \in I)$ under the isomorphisms in Theorem 7.6.1. The above analysis shows that we have the lemma.

Lemma 7.6.3. *The following diagram commutes:*

$$\begin{array}{ccc} (M_{\lambda^2} \otimes V_{\lambda^1}) \otimes (M_{\lambda^2} \otimes V_{\lambda^1}) & \xrightarrow{(,)} & \mathbb{Q}(v) \\ \downarrow & & \| \\ \mathcal{K}_{D^\bullet} \times \mathcal{K}_{D^\bullet} & \xrightarrow{(,)} & \mathbb{Q}(v), \end{array}$$

where the vertical map is induced from (68).

Similar to [L93, Theorem 14.2.3], we have

Theorem 7.6.4. *For any element $K \in \mathcal{K}_{D^\bullet}$, $\pm K \in \mathcal{P}_{D^\bullet}$ if and only if K satisfies the following conditions:*

$$K \in {}_{\mathbb{A}}\mathcal{K}_{D^\bullet}, \quad \mathbf{D}(K) = K, \quad \textit{and} \quad (K, K) \in 1 + v^{-1}\mathbb{Z}[[v^{-1}]].$$

We can use the module $M_{\lambda^1} \otimes V_{\lambda^2}$ and its involution Ψ to define the so-called *based module*, $(M_{\lambda^1} \otimes V_{\lambda^2}, P_\diamond)$ in an algebraic way similar to [L93, 27.1.2, 27.3], where we need to use the notion of crystal basis given in [K91, 3.5] . The basis $P_\diamond$ is called the *canonical basis* (or *global crystal base*) of $M_{\lambda^1} \otimes V_{\lambda^2}$. Moreover, by Corollary 7.6.2, Lemma 7.6.3 and Theorem 7.6.4, we have

Theorem 7.6.5. *The pair $(\mathcal{K}_{D^\bullet}, \mathcal{P}_{D^\bullet})$ together with the involution $\mathbf{D}$ is isomorphic to the based module $(M_{\lambda^1} \otimes V_{\lambda^2}, P_\diamond)$ with associated involution Ψ.*

Remark 7.6.6. (1). The two spaces $\mathcal{K}_{D^\bullet}$ and $\mathbf{K}(\mathbf{d}^\bullet)$ are not the same in general, except when $\lambda^2 = 0$ or $\lambda^1 = 0$.

(2). It is very interesting to interpret naturally the E_i-action as a complex of functors by using (66) or the expression in Lemma 3.1.2.

(3). The data $(M_{\lambda^2} \otimes V_{\lambda^1}, \phi_i, \epsilon_i)_{|i \in I}$ is a module of Kashiwara's *reduced q-analogue* in [K91].

(4). Note that the modules $\mathcal{K}_{D^\bullet}$ are projective modules in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of the quantum group $\mathbf{U}$ in [BGG76, H08, AM11]. We are very close to recover all indecomposable projectives P_λ with $\lambda \in \mathbf{X}$ in $\mathcal{O}$ from this geometric setting.

7.7. **Stability conditions for $\mathcal{Q}_{V,D^\bullet}$.** Recall from Section 7.1, $\mathcal{N}_{V,D^\bullet}$ is the full subcategory of $\mathcal{Q}_{V,D^\bullet}$ generated by the simple objects K satisfying the local stability condition (53):

$$\mathrm{Supp}(\Phi_{\Omega}^{\Omega_i} K) \subseteq \mathbf{E}_{\Omega_i, i, \geq 1}(V, D), \quad \text{for some } i \text{ in } I.$$

One can show that the condition is equivalent to the condition that the complex K is a direct summand, up to a shift, of the complex $L_{(\mathbf{i}^1,\mathbf{a}^1)d^1(\mathbf{i}^2,\mathbf{a}^2)(i,d_i^2+1)d^2}$ for some $i \in I$.

Similar to $\mathcal{M}_{V,D}$ in Section 6.3, we define $\mathcal{M}_{V,D^\bullet}$ to be the full subcategory of $\mathcal{Q}_{V,D^\bullet}$ consisting of all complexes K satisfying the following global stability condition:

$$\mathrm{SS}(K) \cap \mathbf{\Pi}^s_{V,D^\bullet} = \emptyset. \tag{69}$$

Theorem 7.7.1. *We have $\mathcal{N}_{V,D^\bullet} = \mathcal{M}_{V,D^\bullet}$.*

Proof. The proof is similar to Proposition 6.3.1. Suppose that K is a simple object in $\mathcal{N}_{V,D^\bullet}$. Then we may assume that K is a direct summand of the complex $L_{(\mathbf{i}^1,\mathbf{a}^1)d^1(\mathbf{i}^2,\mathbf{a}^2)(i,d_i^2+1)d^2}$ for some $i \in I$. By Proposition 7.3.2, $\mathrm{SS}(K) \subseteq \tilde{Y}_{(\mathbf{i}^1,\mathbf{a}^1)d^1(\mathbf{i}^2,\mathbf{a}^2)(i,d_i^2+1)d^2}$ defined in Section 7.3. Suppose that X is an element in the latter variety. Then it is clear that $X(i)$ is not injective. Thus $\tilde{Y}_{(\mathbf{i}^1,\mathbf{a}^1)d^1(\mathbf{i}^2,\mathbf{a}^2)(i,d_i^2+1)d^2} \cap \mathbf{\Pi}^s_{V,D^\bullet} = \emptyset$. So we have $K \in \mathcal{M}_{V,D^\bullet}$, i.e., $\mathcal{N}_{V,D^\bullet} \subseteq \mathcal{M}_{V,D^\bullet}$. Moreover, the following sets have the same cardinality $\sum_{\nu^1+\nu^2=\nu} \dim V_{\lambda^1,\nu^1} \otimes T_{\lambda^2,\nu^2}$:

the set, say S_1, of isomorphism classes of simple perverse sheaves in $\mathcal{N}_{V,D^\bullet}$;

the set, say S_2, of irreducible components in $\mathbf{\Pi}_{V,D^\bullet}$ disjoint from $\mathbf{\Pi}^s_{V,D^\bullet}$.

Indeed, the fact that $\#S_1 = \#S_2$, can be proved in a similar way as that of $Y_\bullet$ is bijective in Proposition 7.4.2 by taking consideration of the fact that if $Y \in S_2$, then Y is in some subvariety of the form $\tilde{Y}_{(\mathbf{i}^1,\mathbf{a}^1)d^1(\mathbf{i}^2,\mathbf{a}^2)(i,d_i^2+1)d^2}$. And the fact that their total numbers of elements are $\sum_{\nu^1+\nu^2=\nu} \dim V_{\lambda^1,\nu^1} \otimes T_{\lambda^2,\nu^2}$ follows from the fact that the total number of irreducible components of $\mathbf{\Pi}^s_{V,D^\bullet}$ is $\sum_{\nu^1+\nu^2=\nu} \dim V_{\lambda^1,\nu^1} \otimes V_{\lambda^2,\nu^2}$ by Propositions 4.7.1 and Theorem 7.6.1.

By Proposition 7.4.2, the assignment $K \mapsto Y_K$ defines a bijection $\phi : S_1 \to S_2$. By using Proposition 7.4.2 and a similar argument in the proof of Proposition 6.3.1, we have $\mathcal{N}_{V,D^\bullet} = \mathcal{M}_{V,D^\bullet}$. The theorem follows. $\square$

Let ${}_{\mathbb{A}}\mathcal{L}_{D^\bullet}$ and $\mathcal{L}_{D^\bullet}$ be the spaces defined similar to ${}_{\mathbb{A}}\mathcal{K}_{D^\bullet}$ and $\mathcal{K}_{D^\bullet}$, respectively, if we replace $\mathcal{Q}_{V,D^\bullet}$ by its quotient category $\mathcal{V}_{V,D^\bullet}$ in Section 7.1. By Theorems 7.6.1 and 7.7.1, we have

Corollary 7.7.2. ${}_{\mathbb{A}}V_{\lambda^2} \otimes {}_{\mathbb{A}}V_{\lambda^1} \simeq {}_{\mathbb{A}}\mathcal{L}_{D^\bullet}$ *and* $V_{\lambda^2} \otimes V_{\lambda^1} \simeq \mathcal{L}_{D^\bullet}$.

Let $\pi_{D^\bullet} : \mathcal{K}_{D^\bullet} \to \mathcal{L}_{D^\bullet}$ be the natural projection. Let

$$\mathcal{B}_{D^\bullet} = \pi_{D^\bullet}(\mathcal{P}_{D^\bullet}) \backslash \{0\}.$$

It is clear by Theorems 7.6.1 and 7.7.1 that $\mathcal{B}_{D^\bullet}$ is a basis of ${}_{\mathbb{A}}V_{\lambda^2} \otimes {}_{\mathbb{A}}V_{\lambda^1}$ under the above isomorphism. Moreover, the involution $\mathbf{D}$ induces an involution $\mathbf{D}$ on $\mathcal{L}_{D^\bullet}$ commuting with the involution Ψ on $V_{\lambda^2} \otimes V_{\lambda^1}$ defined in [L93, 27.3.1] and the crystal structure on $\mathcal{P}_{D^\bullet}$ induces a crystal structure on $\mathcal{B}_{D^\bullet}$.

Corollary 7.7.3. *$(\mathcal{L}_{D^\bullet}, \mathcal{B}_{D^\bullet})$ with the associated involution $\mathbf{D}$ is a based module in the sense of* [L93, 27.1.2]*, isomorphic to $(V_{\lambda^2} \otimes V_{\lambda^1}, B_\diamond)$, where $B_\diamond$ is the canonical basis of $V_{\lambda^2} \otimes V_{\lambda^1}$.*

The above analysis shows that the pair $(\mathcal{L}_{D^\bullet}, \mathcal{B}_{D^\bullet})$ satisfies the conditions [L93, 27.1.2, (a)-(c)]. The condition [L93, 27.1.2, (d)] can be proved by a result for $\mathcal{P}_{D^\bullet}$ analogous to [L93, Proposition 18.1.7], which is left to the reader. The identification of $\mathcal{B}_{D^\bullet}$ with $B_\diamond$ is due to a similar diagram as that in Corollary 7.6.2.

Remark 7.7.4. Note that Corollaries 7.7.2 and 7.7.3 are first proved in [Zh08] by a different method.

The above results implies that the second row in (61) is a categorical version of the exact sequence $0 \to T_{\lambda^2} \otimes V_{\lambda^1} \to M_{\lambda^2} \otimes V_{\lambda^1} \to V_{\lambda^2} \otimes V_{\lambda^1} \to 0$.

8. General case

The results in Section 7 can be generalized directly to the tensor product of N copies of irreducible integrable representations. We will state the analogous results in this section. The results can be proved inductively or by mimicking the one in the $N = 2$ case.

8.1. **Results on $\mathbf{\Pi}_{V,D^\bullet}$.** We fix $\lambda^1, \cdots, \lambda^N$ in $\mathbf{X}$, $d^1, \cdots, d^N$ in $\mathbb{N}[I]$ and $D^1, \cdots, D^N$ such that

$$\lambda = \lambda^1 + \cdots + \lambda^N, \quad d = d^1 + \cdots + d^N, \quad \text{and} \quad D = D^N \oplus \cdots \oplus D^1,$$

and $d_i^a = \lambda^a(i) = \dim D_i^a$, for any $i \in I$ and $a = 1, \cdots, N$. Let

$$D^\bullet = (\check{D}^0 \supset \check{D}^1 \supset \cdots \supset \check{D}^N)$$

be the flag with $\check{D}^a = \oplus_{b=a+1}^N D^b$ for $a = 0, \cdots, N$. The tensor product variety $\mathbf{\Pi}_{V,D^\bullet}$ is the closed subvariety of $\mathbf{\Lambda}_{V,D}$ consisting of all nilpotent elements $X = (x, p, q)$ such that

$$\overline{p(\check{D}^a)} \subseteq \underline{q^{-1}(\check{D}^{a+1})} \quad \forall a = 0, \cdots, N-2 \quad \text{and} \quad p(\check{D}^{N-1}) = 0. \tag{70}$$

When $N = 2$, this is the same as the one defined in (26).

One can prove inductively the following results.

Theorem 8.1.1. *The following statements hold.*

(1) $\mathbf{\Pi}_{V,D^\bullet}$ *has pure dimension* $\frac{1}{2}\dim \mathbf{E}(V, D)$*. Moreover,*

$$\#\mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet} = \sum_{V^1,\cdots,V^N} \#\mathrm{Irr}\, (\mathbf{L}^s_{V^1,D^1} \times \cdots \times \mathbf{L}^s_{V^{N-1},D^{N-1}} \times \mathbf{L}_{V^N,D^N}),$$

where the sum runs over the representatives $(V^1, \cdots, V^N)$ *of* $(\nu^1, \cdots, \nu^N)$ *such that* $\nu^1 + \cdots + \nu^N = \nu = \dim V$*.*

(2) *The set* $\mathrm{Irr}\,(D^\bullet) = \sqcup_V \mathrm{Irr}\, \mathbf{\Pi}_{V,D^\bullet}$*, equipped with a crystal structure defined similar to that in Section 4.6, is isomorphic to the crystal* $B(\lambda^1) \otimes \cdots \otimes B(\lambda^{N-1}) \otimes B(\lambda^N, \infty)$*.*

Note that if we consider the class of constructible functions on $\mathbf{\Pi}_{V,D^\bullet}$, similar to the complexes $L_{\underline{\mathbf{a}}}$, we will get a geometric realization of the module of the enveloping algebra U associated with the graph Γ, similar to the module $M_{\lambda^N} \otimes V_{\lambda^{N-1}} \otimes \cdots \otimes V_{\lambda^1}$. The E_i-action can be defined in a way similar to that of M_λ in [GLS06]. A basis for this U-module can also be obtained as the semicanonical basis for U^- in [L00a].

8.2. **Results on $\mathcal{P}_{V,D^\bullet}$.** Let

$$\underline{\mathbf{a}} := (\mathbf{i}^1, \mathbf{a}^1) \cdot d^1 \cdots (\mathbf{i}^{N-1}, \mathbf{a}^{N-1}) \cdot d^{N-1} \cdot (\mathbf{i}^N, \mathbf{a}^N) \cdot d^N.$$

We can define the object $\mathcal{Q}_{V,D^\bullet}$, $\mathcal{P}_{V,D^\bullet}$, $\mathcal{P}_{D^\bullet}$, $\mathcal{N}_{V,D^\bullet}$, $\mathcal{M}_{V,D^\bullet}$, $\mathcal{V}_{V,D^\bullet}$ $\mathcal{B}_{D^\bullet}$ and $\mathcal{K}_{D^\bullet}$ in exactly the same manner as those in Section 7. Moreover, the results in Section 7 are still true with minor modifications. The proofs are similar to those in $N = 2$ cases by taking into consideration of Proposition 8.1.1. In particular, we have

Theorem 8.2.1. (1) *The set $\mathcal{P}_{D^\bullet}$, together with the crystal structure defined similar to the one in Section 7.4, is isomorphic to the crystal $B(\lambda^1)\otimes\cdots\otimes B(\lambda^{N-1})\otimes B(\lambda^N,\infty)$.*

(2) *For any $K \in \mathcal{P}_{VD^\bullet}$, its singular support satisfies a similar relation as in Proposition 7.4.2.*

(3) *$(\mathcal{K}_{D^\bullet}, \mathcal{P}_{D^\bullet})$ is isomorphic to the based module $(M_{\lambda^N} \otimes V_{\lambda^{N-1}} \otimes V_{\lambda^{N-2}} \otimes \cdots \otimes V_{\lambda^1}, P_\diamond)$, where $P_\diamond$ is the canonical basis of $M_{\lambda^N} \otimes V_{\lambda^{N-1}} \otimes V_{\lambda^{N-2}} \otimes \cdots \otimes V_{\lambda^1}$.*

(4) *$\mathcal{N}_{V,D^\bullet} = \mathcal{M}_{V,D^\bullet}$.*

(5) *The based module $(\mathcal{L}_{D^\bullet}, \mathcal{B}_{D^\bullet})$ is isomorphic to $(V_{\lambda^\bullet}, B_\diamond)$ where $B_\diamond$ is the canonical basis of $V_{\lambda^\bullet}$.*

Note that (5) has been proved in [Zh08] by a different method.

9. Acknowledgement

We thank Catharina Stroppel for her invitation to University of Bonn, where this work was originated and sending us the paper [FSS11]. We thank Wei Liang Gan for his invitation to present this work in the workshop of Lie groups, Lie algebras and their representations, University of California, Riverside, May 21-22, 2011. We also thank Igor B. Frenkel, Zongzhu Lin, Raphael Rouquier, Mark Shimozono, Josh Sussan and Ben Webster for helpful conversations. We thank Jim Humphreys for sending the author a copy of [H77]. We thank Yoshiyuki Kimura for pointing out a mistake in a previous version of this paper and sending us the paper [K07].

Partial support from NSF grant DMS 1101375 / 1160351 is acknowledged. Part of the work was done while the author was in Virginia Tech.

References

[AM11] H. Andersen, V. Mazorchuk, *Category $\mathcal{O}$ for quantum groups*, arXiv:1105.5500.

[BBD82] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982).

[BGG71] I. N. Bernstein, I. M. Gelfand, I. S. Gelfand, *Structure of representations generated by vectors of highest weight.* Functional Anal. Appl. **5** (1971), 1-8.

[BGG75] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *Differential Operators on the Base Affine Space and a Study of $\mathfrak{g}$-Modules*, Lie Groups and their representations (New York) (I. M. Gelfand, ed.), Halsted Press, 1975, pp. 21-64.

[BGG76] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *A certain category of $\mathfrak{g}$-modules.* (Russian) Funkcional. Anal. i Prilozen. 10 (1976), no. 2, 1-8.

[BG80] J. N. Bernstein, S.I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras.* Compositio Math. **41** (1980), no. 2, 245-285.

[FSS11] I. Frenkel, C. Stroppel, J. Sussan, *Categorifyig fractional Euler characteristics, Jones-Wenzl Projector and $3j$-symbols*, arXiv:1007.4680.

[GLS06] C. Geiss, B. Leclerc and J. Schroer, *Verma modules and preprojective algebras*, Nagoya Math J. **182** (2006), 241-258.

[H77] J. Humphreys, *A construction of projective modules in the category $\mathcal{O}$ Bernstein-Gelfand-Gelfand.* Nederl. Akad. Wetensch. Proc. Ser. A 80=Indag. Math. **39** (1977), no. 4, 301-303.

[H08] J. Humphreys, *Representations of semisimple Lie algebras in the BGG category O.* Graduate Studies in Math. **94**. American Mathematical Society, Providence, RI, 2008.

[K91] M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras.* Duke Math. J. **63** (1991), no. 2, 465-516.

[KS90] M. Kashiwara, P. Schapira, *Sheaves on manifolds. With a chapter in French by Christian Houzel.* Fundamental Principles of Mathematical Sciences, **292**. Springer-Verlag, Berlin, 1994.

[KS97] M. Kashiwara, Y. Saito, *Geometric construction of crystal bases.* Duke Math. J. **89** (1997), no. 1, 936.

[K07] Y. Kimura, *Affine quivers and crystal bases*, master thesis, 2007.

[L90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447-498.

[L90b] G. Lusztig, *Canonical bases arising from quantized enveloping algebras. II. Common trends in mathematics and quantum field theories (Kyoto, 1990).* Progr. Theoret. Phys. Suppl. No. **102** (1990), 175-201 (1991).

[L91] G. Lusztig, *Quivers, Perverse Sheaves, and Quantized Enveloping Algebras*, J. Amer. Math. Soc., **4** (1991), 365-421.

[L92] G. Lusztig, *Canonical bases in tensor products.* Proc. Nat. Acad. Sci. U.S.A. **89** (1992), no. 17, 8177-8179.

[L93] G. Lusztig, *Introduction to Quantum Groups*, Progress in Math. **110**, Birkhäuser 1993.

[L00a] G. Lusztig, *Semicanonical bases arising from enveloping algebras.* Adv. Math. **151** (2000), no. 2, 129-139.

[M03] A. Malkin, *Tensor product varieties and crystals: the ADE case.* Duke Math. J. **116** (2003), no. 3, 477-524.

[N94] H. Nakajima, *Instantons on ALE spaces, quiver varieties and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365-416.

[N98] H. Nakajima, *Quiver varieties and Kac-Moody algebras.* Duke Math. J. **91** (1998), no. 3, 515-560.

[N01] H. Nakajima, *Quiver varieties and tensor products.* Invent. Math. **146** (2001), no. 2, 399-449.

[S02] Y. Saito, *Crystal bases and quiver varieties.* Math. Ann. **324** (2002), no. 4, 675-688.

[Sch09] O. Schiffmann, *Lectures on canonical and crystal bases of Hall algebras*, arXiv:0910.4460.

[Zh08] H. Zheng, *Categorification of integrable representations of quantum groups*, arXiv:0803.3668.

Department of Mathematics, University at Buffalo, State University of New York, 244 Mathematics Building, Buffalo, New York 14260

E-mail address: `yiqiang@buffalo.edu`